STOCHASTIC ANALYSIS.

winter term 2015, Jiří Černý.

Chapter I. INTRODUCTION

The objective of this course is to construct Brownian motion, present its most important properties and develop the infinitesimal calculus attached to it. We will also see that Brownian motion is an element of various important classes of processes, e.g. it is

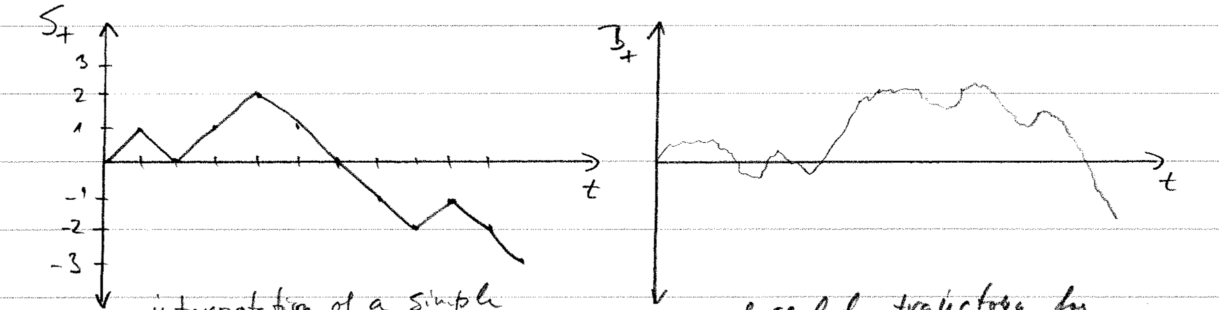
- continuous martingale
- continuous Markov process
- Gaussian process
- Lévy process.

We will thus use the Brownian motion as a pretext to give an introduction into the mathematical theory of these classes.

History. The Brownian motion (BM) is named after the botanist Robert Brown, who in 1827 observed the irregular motion of pollen particles suspended in water. First mathematical studies of BM appear on the turn of XX. century. In particular Bachelier in his thesis from 1900 used the BM as a model for a stock market, and, in 1905, Einstein considered the BM in his fundamental work on the movement of particles in a fluid, giving one of the first (indirect) confirmation of the existence of atoms and molecules. The mathematical theory of BM was then put on a firm basis with Wiener (1923)

Informal construction.

There are many ways how to construct the BM. From previous lectures you probably know the, probably, most natural one, as the scaling limit of a (polygonal interpolation of) random walk trajectory, scaling the time by n^{-2} and space by n^{-1}:



interpolation of a simple random walk trajectory

rescaled trajectory for large n

Let $X_1, X_2, \dots$ be iid Bernoulli r.v., $P(X_i=-1)=P(X_i=1)=\frac{1}{2}$. Set $S_n = X_1+\dots+X_n$, $n\in\mathbb{N}$, and define S_t, $t\in\mathbb{R}_+$, by linear interpolation (see Figure). Finally, define the rescaled trajectory

(1.1) $$B_t^{(n)} = \frac{1}{n} S_{n^2 t}, \quad t\geq 0.$$

From the central limit theorem, $B_1^{(n)} \xrightarrow[n\to\infty]{law} \mathcal{N}(0,1)$, a standard normal r.v., i.e. $\lim_{n\to\infty} P(B_1^{(n)}\leq a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}\,dx$, $a\in\mathbb{R}$.
Donsker's theorem (1951) then shows that $B_\cdot^{(n)}$ viewed as a random continuous function on $\mathbb{R}_+$ converges (locally in the uniform topology) to the law of the BM.

We will see other construction(s) in this lecture.

An important advantage of continuous models (like BM) vs. discrete models (like RW) is the apparatus of "infinitesimal calculus". The development of this apparatus is rather non-trivial because the typical realisation of the BM trajectory is rather rough: the function $t\mapsto B_t(\omega)$ is continuous but nowhere differentiable and of infinite variation.

As an example, consider the basic formula of calculus:

(1.2) $$\frac{d}{dt} f(b(t)) = f'(b(t))\, b'(t), \qquad f, b \in C^1$$

It can be extended to $f \in C^1$ and b continuous of finite variation by writing

(1.3) $$f(b(t)) = f(b(0)) + \int_0^t f'(b(s))\, db(s), \qquad t \geq 0,$$

where db stands for the Stieltjes measure on $\mathbb{R}_+$ given by

(1.4) $$\int_{(x,y]} db(s) = b(y) - b(x), \qquad 0 \leq x < y < \infty.$$

As the BM is not of finite variation, this does not help when b is a BM trajectory.

Nonetheless, we will develop an infinitesimal calculus where (1.3) get replaced by the so-called Itô's formula:

(1.5) $$f(B_t) = f(B_0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2}\int_0^t f''(s)\, ds, \quad f \in C^2, t \geq 0.$$

or, in differential notation of (1.2)

(1.6) $$df(B_t) = f'(B_t)\, dB_t + f''(B_t)\, dt.$$

The main work will be to give a sense to the term "$\int_0^t f'(B_s) dB_s$" appearing as (1.5), since it cannot be defined as "Stieltjes integral", as explained.

With this calculus we will be able to solve certain differential equations with random perturbations, the "stochastic differential equations", SDE's

(1.7) $$dX_t = b(X_t)\,dt + \overbrace{\sigma(X_t)\, dB_t}^{\text{random perturbation}}$$

We will also see the deep connection between SDE's and certain PDE's. As example, consider $D \subset \mathbb{R}^d$ to be a smooth bounded domain, and the following two PDE problems:

Dirichlet problem: given $f \in \partial D$, find u such that

(1.8) $$\frac{1}{2}\Delta u = 0 \text{ in } D \quad ; \quad u|_{\partial D} = f.$$

Poisson problem: for $g \in C(\bar{D})$, find u such that

(1.9) $$\frac{1}{2}\Delta u = g \text{ in } D \quad ; \quad u|_{\partial D} \equiv 0.$$

Surprisingly, both these problems can "be solved" with help of BM.
To this end let $B_t = (B_t^1, \dots, B_t^d)$ be the d-dimensional BM
(i.e. $B^1, \dots, B^d$ are independent copies of the real-valued BM)
For $x \in D$, define

$$\tau_x = \inf \{ s \geq 0 : x + B_s \in \partial D \}$$

Then the solutions to (1.8), (1.9) are given by

(1.10) $$u_{\text{Dirichlet}}(x) = E\left[f(x + B_{\tau_x}) \right]$$

(1.11) $$u_{\text{Poisson}}(x) = -E\left[\int_0^{\tau_x} g(x + B_s)\, ds \right].$$

We will also be able to replace $\frac{1}{2}\Delta$ in (1.8), (1.9) by more complicated differential operators.

Literature: see the lecture webpage at
www.mat.univie.ac.at/~cerny/teaching.html.

Chapter II. GAUSSIAN VECTORS & PROCESSES.

In the next chapter we will define the BM as a particular "Gaussian process". We will give an introduction to the theory of these processes in this chapter.

2.1 One-dimensional Gaussian distribution.

We recall that a r.v. X is called standard normal or standard Gaussian if it has a density on $\mathbb{R}$ given by
$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

(2.1) Exercise: (a) Show that the Laplace transform of X is given by
$$E[e^{zX}] = e^{z^2/2} \quad \text{for every } z \in \mathbb{C}.$$
(Hint: Consider first $z \in \mathbb{R}$ and prove and use the analyticity).
In particular, show that the characteristic function of X is
$$E[e^{i\xi X}] = e^{-\xi^2/2}, \quad \xi \in \mathbb{R}.$$
(b) Using the properties of characteristic functions show that $X \in L^p$ (i.e. $E[|X|^p] < \infty$) and that
$$E[X] = 0, \quad E[X^2] = 1, \quad E[X^{2n}] = \frac{(2n)!}{2^n n!}$$

For $m \in \mathbb{R}$, $\sigma > 0$, we say that r.v. Y is $\mathcal{N}(m, \sigma^2)$-distributed if Y can be written as

(2.2) $$Y = m + \sigma X \quad \text{where } X \text{ is standard normal r.v.}$$

(2.3) Exercise: (a) Show that (2.2) is equivalent with
(i) Y has a density $p_Y(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$
(ii) the char. function of Y is $E[e^{i\xi Y}] = e^{im\xi - \frac{\sigma^2}{2}\xi^2}$
(b) Prove $E[Y] = m$, $Var[Y] = \sigma^2$.
(c) Show: Let Y be $\mathcal{N}(m_Y, \sigma_Y^2)$ distributed, Z be $\mathcal{N}(m_Z, \sigma_Z^2)$ distributed and Y be independent of Z. Then $Y+Z$ is $\mathcal{N}(m_Y + m_Z, \sigma_Y^2 + \sigma_Z^2)$-distributed

(2.4) <u>Lemma</u>: Let X_n be a sequence of r.v.'s, $X_n \sim \mathcal{N}(m_n, \sigma_n^2)$. Assume that X_n converge in distribution to a r.v. X. Then
(a) X is also Gaussian r.v, $X \sim \mathcal{N}(m, \sigma^2)$, $m = \lim_n m_n$, $\sigma = \lim_n \sigma_n$
(b) If X_n converges in probability, then it converges in L^p, $p \in [1,\infty)$

<u>Proof</u>: (a) Convergence in distribution is equivalent with
$$E[e^{i\xi X_n}] = \exp\{i m_n \xi - \frac{\sigma_n^2}{2}\xi^2\} \xrightarrow[n\to\infty]{} E[e^{i\xi X}], \forall \xi \in \mathbb{R}$$
Taking abs. values, $\exp\{-\frac{\sigma_n^2}{2}\xi^2\} \xrightarrow[n\to\infty]{} |E[e^{i\xi X}]|$, which holds only if $\sigma_n \to \sigma \geq 0$ ($\sigma = \infty$ can be excluded, since $E[e^{i\xi X}]$ must be continuous at 0). Hence, for all $\xi \in \mathbb{R}$

(2.5) $$e^{i m_n \xi} \xrightarrow[n\to\infty]{} e^{\frac{1}{2}\sigma^2\xi^2} E[e^{i\xi X}].$$

To prove that (2.5) implies $\lim_n m_n = m$ observe first that m_n must be a bounded sequence. Indeed, assume by contradiction that for some $n_k \nearrow \infty$ we have $m_{n_k} \nearrow \infty$. Then for every $A > 0$, $P[X \geq A] \geq \limsup_{k\to\infty} P[X_{n_k} \geq A] \geq \frac{1}{2}$. Letting $A \nearrow \infty$ we obtain a contradiction. Assume now that m_n is bounded and does not converge. Then it has at least two accumulation points $m \neq m'$ and by (2.5) $e^{i\xi m} = e^{i\xi m'}$ $\forall \xi \in \mathbb{R}$. But this is possible only if $m = m'$.

– (b) X_n can be written as $X_n = m_n + \sigma_n X$ with $X \sim \mathcal{N}(0,1)$. By (a), m_n and σ_n are bounded sequences which implies $\sup_n E[|X_n|^q] < \infty$ $\forall q \in (0,\infty)$. By Fatou's lemma then also $E[|X|^q] < \infty$.

Let now $p \in [1,\infty)$. By assumption, $Y_n = |X_n - X|^p$ converges to 0 in probability. Using $q = 2p$, we see from the previous reasoning that Y_n is bounded in L^2 and so uniformly integrable. Hence $Y_n \to 0$ in L^1, proving $X_n \to X$ in L^p. □

2.2 Gaussian vectors:

Let E be a d-dimensional Euclidean space with scalar product $\langle \cdot , \cdot \rangle$.

(2.6) Definition: E-valued r.v. X is called Gaussian vector if $\langle u, X\rangle$ is a Gaussian r.v. for every $u \in E$.

(2.7) Example: $E = \mathbb{R}^d$, $X_1, \dots, X_d$ independent Gaussian r.v's. Then $X = (X_1, \dots, X_d)$ is a Gauss. vector.

(2.8) Exercise Use (2.3(c)) to show this claim.

If X is a Gaussian vector in E, then there exist $m_X \in E$ and positive quadratic form $q_X : E \to \mathbb{R}$ such that

$$E[\langle u, X\rangle] = \langle u, m_X\rangle$$
$$\operatorname{Var}[\langle u, X\rangle] = q_X(u).$$

(2.9) Exercise Prove the above claim. To this end consider an orthonormal basis $(e_1, \dots, e_d)$ of E and show that if $X = X_1 e_1 + \dots + X_d e_d$ with $X_i = \langle X, e_i\rangle$ then $m_X = \sum_{i=1}^{d} E[X_i]\, e_i =: EX$ and for $u = \sum_{i=1}^{d} u_i e_i$

$$q_X(u) = \sum_{i,j=1}^{d} u_i u_j \operatorname{Cov}(X_i, X_j).$$

Show that the characteristic function of X is

$$E[e^{i\langle \xi, X\rangle}] = \exp\{ i\langle m_X, \xi\rangle - \tfrac{1}{2} q_X(\xi)\}, \quad \xi \in E.$$

(2.10) Exercise: The coordinates $(X_1, \dots, X_d)$ of a Gauss. vector X are independent iff the matrix $(\operatorname{Cov}(X_i, X_j))_{i,j=1,\dots,d}$ is diagonal, that is q_X is diagonal in the basis $(e_1, \dots, e_d)$.

To q_X is associated a positive symmetric endomorphism γ_X via

$$q_X(u) = \langle u, \gamma_X(u) \rangle.$$

(γ_X has matrix $(\mathrm{Cov}(X_i, X_j))_{i,j}$ in the basis $(e_1, \dots, e_d)$)

‖ Starting from here we only consider centred G. vectors, $m_X = 0$ ‖

(2.11) Theorem: (Existence & properties of G. vectors).

(a) For every positive symmetric endomorphism γ on E there is a G. vector X s.t. $\gamma = \gamma_X$

(b) Let X be a G. vector. If $(\varepsilon_1, \dots, \varepsilon_d)$ is the basis diagonalising γ_X: $\gamma_X \varepsilon_j = \lambda_j \varepsilon_j$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0 = \lambda_{r+1} = \dots = \lambda_d$. Then

$$X = \sum_{i=1}^{r} Y_j \varepsilon_j$$

where Y_i, $i = 1, \dots, r$, are independent, $Y_i \sim \mathcal{N}(0, \lambda_i)$. In particular

$$\mathrm{supp}\,(\text{law of } X) = \mathrm{span}\,(\varepsilon_1, \dots, \varepsilon_r)$$

(c) Law of X is abs. continuous w.r.t. Lebesgue measure on E iff $r = d$. In this case X has a density

$$p_X(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det \gamma_X}} \exp\left\{-\tfrac{1}{2}\langle x, \gamma_X^{-1}(x)\rangle\right\}.$$

Proof: (a) Let $(\varepsilon_1, \dots, \varepsilon_d)$ be an orthonormal basis diagonalizing γ, i.e. $\gamma \varepsilon_i = \lambda_i \varepsilon_i$, and Y_i independent, $Y_i \sim \mathcal{N}(0, \lambda_i)$. Set

$$X = \sum_{i=1}^{d} Y_i \varepsilon_i$$

Then one sees easily that for $u = \sum u_i \varepsilon_i$

$$q_X(u) = E\Big[\Big(\sum_{i=1}^{d} u_j Y_j\Big)^2\Big] = \sum \lambda_j u_j^2 = \langle u, \gamma u \rangle$$

– (b). By definition of the basis $(\varepsilon_1, \dots, \varepsilon_d)$, the covariance matrix $\mathrm{Cov}(Y_i, Y_j) = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$. For $i > r$, $E[Y_i^2] = 0$, so $Y_i = 0$ a.s. Using then (2.10) we see that $(Y_i)_{i \leq r}$ are independent. Since $X = \sum_{i=1}^{r} Y_i \varepsilon_i$ a.s., then $\mathrm{supp}\,(\text{law of } X) \subset \mathrm{span}\,(\varepsilon_1, \dots, \varepsilon_r)$. On the other hand, $P\left[X \in \prod_{i=1}^{r}[a_i, b_i]e_i\right] = \prod_{i=1}^{r} P[Y_i \in [a_i, b_i]] > 0$, we see that $\mathrm{supp}\,(\text{law of } X) \supset \mathrm{span}\,(\varepsilon_1, \dots, \varepsilon_r)$.

– (c). If $r<d$, supp(law of X) $\in$ span$(\varepsilon_1, \dots, \varepsilon_r)$ which has 0 Lebesgue measure on E. If $r=d$, let $Y=(Y_1, \dots, Y_d)$ be a random vector in $\mathbb{R}^d$. X is then image of Y by a bijection $\varphi: \mathbb{R}^d \to E$ given by $\varphi(y_1, \dots, y_d) = \sum_{i=1}^d y_i \varepsilon_i$. Then

$$E[g(X)] = E[g(\varphi(Y))] \overset{d}{=}$$

$$\overset{(2.3(i))}{=} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} g(\varphi(y)) \prod_{i=1}^d \frac{dy_i}{\sqrt{\lambda_i}} \exp\left\{-\frac{1}{2}\frac{y_i^2}{\lambda_i}\right\}$$

$$= \frac{1}{(2\pi)^{d/2}} \frac{1}{\sqrt{\det \gamma_X}} \int_{\mathbb{R}^d} g(\varphi(y)) \exp\left\{-\frac{1}{2}\langle \varphi(y), \gamma_X^{-1}\varphi(y)\rangle\right\} dy_1 \dots dy_d$$

$$= \frac{1}{(2\pi)^{d/2}} \frac{1}{\sqrt{\det \gamma_X}} \int_E g(x) \exp\left\{-\frac{1}{2}\langle x, \gamma_X^{-1} x\rangle\right\} dx$$

where in the last equality we used the fact that the Lebesgue measure on E is the image of the Lebesgue measure on $\mathbb{R}^d$ by φ. $\square$.

2.3. Gaussian spaces & processes

(2.12) Definition: Gaussian space is a closed sub-space of the space $L^2(\Omega, \mathcal{A}, P)$ formed by centred Gaussian r.v.'s.

(2.13) Example: If $(Y_1, \dots, Y_d) \in \mathbb{R}^d$ is a Gaussian vector, then $\overline{\text{span}}(X_1, \dots, X_d)$ is a Gaussian space.

(2.14) Definition. A collection $X=(X_t)_{t\in T}$ of real-valued r.v's on some $(\Omega, \mathcal{A}, P)$ is called Gaussian process if every finite linear combination of X_t, $t\in T$, is a (centred) Gaussian r.v.

(2.15) Lemma. If $(X_t)_{t\in T}$ is a G. process, then the closed subspace of $L^2(\Omega, \mathcal{A}, P)$ generated by r.v. X_t, $t\in T$, is a G. space, which is called G. space generated by X

Proof: It is sufficient to observe that a limit in L^2 of G. r.v.'s is again a G. r.v., by Lemma (2.4). $\square$

Existence of Gaussian processes

Let $(X_t)_{t\in T}$ be a G. process. Its covariance function $\Gamma: T\times T\to\mathbb{R}$ is given by

$$\Gamma(s,t) = \mathrm{Cov}(X_s, X_t) = E[X_s X_t], \quad t,s\in T.$$

The distribution of X is determined by Γ. (In the non-centred case one needs in addition a function $m(t) = EX_t$)

Indeed, for every finite collection $\{t_1,\dots,t_k\}\subset T$, the vector $(X_{t_1},\dots,X_{t_k})$ has a G. distribution with covariance matrix $(\Gamma(t_i,t_j))_{i,j=1,\dots,k}$.

On the other hand, one may ask if every $\Gamma: T\times T\to\mathbb{R}$ is a covariance of a G. process. Obviously, Γ needs to be symmetric, $\Gamma(s,t)=\Gamma(t,s)$, and positively definite in the following sense: For every $k\ge 0$, $\{t_1,\dots,t_k\}\subset T$ and $\xi\in\mathbb{R}^k$

(2.16) $$\sum_{i,j=1}^{k} \xi_i \Gamma(t_i,t_j)\xi_j = E\Big[\Big(\sum_{i=1}^{k}\xi_i X_{t_i}\Big)^2\Big] \ge 0.$$

It turns out that those two conditions are sufficient.

(2.17) Theorem: If $\Gamma: T\times T\to\mathbb{R}$ is symmetric and positively definite, then there is a Gaussian process on T with covariance Γ.

Proof: is an easy application of Kolmogorov's extension theorem. We construct a probability P on the measurable space $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^{\otimes T})$ such that the canonical (coordinate) process $X_t(\omega) = \omega_t$ is a G. process with covariance Γ. To this end, let for every $\{t_1,\dots,t_k\}\subset T$, $P_{\{t_1,\dots,t_k\}}$ be the law of G. vector on $\mathbb{R}^{\{t_1,\dots,t_k\}} \cong \mathbb{R}^k$ with covariance $(\Gamma(t_i,t_j))_{i,j=1,\dots,k}$, existing by (2.11). It is easy to check that $P_{\{t_1,\dots,t_k\}}$ satisfy the assumptions of Kolmogorov's extension thm, which yield the existence $\square$

It is, however, rather difficult to check that Γ is positive definite. We give two important examples where it is possible:

(2.18) <u>Example</u>: Let E be countable and $(Z_t)_{t\in\mathbb{R}_+}$ a Markov chain on E with transition probabilities $P[Z_t=y \mid Z_0=x] = p_t(x,y)$ such that $p_t(x,y)=p_t(y,x)$. Consider its Green function

$$g(x,y) = E_x\left[\int_0^\infty 1\{Z_t=y\}\right] = \int_0^\infty p_t(x,y)\,dt < \infty.$$

Then, for every $\xi \in \mathbb{R}^E$ with finite support

$$\sum_{x,y\in E} \xi_x\, g(x,y)\,\xi_y = \sum_{x,y\in E} \int_0^\infty \xi_x\, p_t(x,y)\,\xi_y\,dt =$$

$$= \sum_{x,y,z\in E} \int_0^\infty \xi_x\, p_{t/2}(x,z)\,p_{t/2}(z,y)\,\xi_y\,dt$$

$$= \int_0^\infty \sum_{z\in E} \Big(\sum_{y\in E} p_{t/2}(y,z)\,\xi_z\Big)^2 > 0.$$

(Infact, E does not need to be countable here.) If Z is a BM or RW, the associated G. process is sometimes called <u>Gaussian free field</u>.

(2.19) <u>Example</u>: Let $T=\mathbb{R}$ and let μ be a finite symmetric measure, i.e. $\mu(A)=\mu(-A)$ on $\mathbb{R}$. Set

(2.20) $$\Gamma(s,t) = \int e^{i\lambda(s-t)}\,\mu(d\lambda).$$

Then, for $\xi \in \mathbb{R}^{\mathbb{R}}$ with finite support

$$\sum_{\mathbb{R}\times\mathbb{R}} \xi_s\,\Gamma(s,t)\,\xi_t = \int \Big|\sum_{T} \xi_s\, e^{i\lambda s}\Big|^2 \mu(d\lambda) \geq 0.$$

I.e, by (2.17), there is a corresponding G. process.

Moreover, as $\Gamma(s,t) = \Gamma(s-t)$, (X_t) is stationary, i.e. $\text{law}(X_{t_1+t},\dots,X_{t_k+t}) = \text{law}(X_{t_1},\dots,X_{t_k})$ $\forall k$ finite, $t_1,\dots,t_k,t\in\mathbb{R}$.

The converse is also true.

(2.21) <u>Theorem</u> (Bochner) If $(X_t)_{t\in\mathbb{R}}$ is stationary G. process then there is a symmetric finite measure μ s.t. (2.20) holds.

<u>Proof</u>: see Rogers, Williams, Thm 24.9. $\square$

<u>Orthogonality & independence</u>

Similarly as in (2.10), there is a connection between orthogonality and independence.

(2.22) <u>Theorem</u>: Let H be a G. space and $(H_i)_{i\in I}$ its subspaces. Then H_i, $i\in I$, are orthogonal iff the σ-algebras $\sigma(H_i)$, $i\in I$, are independent.

(2.23) <u>Remark</u>: It is crucial that H_i's are subspaces of the same G. space. To see that, let $X\sim\mathcal{N}(0,1)$ and ε be an independent Bernoulli r.v. $P(\varepsilon=\pm 1)=\frac{1}{2}$. Set $Y=\varepsilon X$. Then $E[XY]=E[\varepsilon]E[X^2]=0$, so X,Y are orthogonal in L^2, but $|X|=|Y|$ so they are not independent. Of course, here (X,Y) is not a G. vector.

<u>Proof</u>: $\Leftarrow$: If $\sigma(H_i)$'s are independent, then for $X_i\in H_i$, $i\neq j$

$$E[X_iX_j]=E[X_i]E[X_j]=0,$$ i.e. H_i's are orthogonal.

$\Rightarrow$: (Sketch - fill the details for exercise).

1. It is sufficient to check that for every finite $\{i_1,\dots,i_p\}\in I$, and random variables $\xi_1^k,\dots,\xi_{j_k}^k\in H_{i_k}$, $1\le k\le p$, the vectors $(\xi_1^k,\dots,\xi_{j_k}^k)$, $1\le k\le p$, are independent. (Use Dynkin's lemma here)

2. Now, for every $k\le p$ find and orthonormal basis $\zeta_1^k,\dots,\zeta_{\tilde{j}k}^k$ of $\operatorname{span}(\xi_1^k,\dots,\xi_{j_k}^k)$. Then the covariance matrix of $(\zeta_1^1,\dots,\zeta_{\tilde{j}_1}^1,\dots,\zeta_1^p,\dots,\zeta_{\tilde{j}_p}^p)$ is the identity matrix, so they are independent by (2.10). This implies the independence required in the first step. $\square$

(2.34) <u>Corollary</u>: Let H be a G. space and K its closed subspace. If p_K denotes the orthogonal projection on K and $X\in H$, then

(i) $E[X|\sigma(K)]=p_K(X)$

(ii) Let $\sigma^2 = E[(X - p_K(X))^2]$. Then for every $\Gamma \in \mathcal{B}(\mathbb{R})$

$$P[X \in \Gamma \mid \sigma(K)] = Q(\omega, \Gamma)$$

where $Q(\omega, \cdot)$ is the distribution $\mathcal{N}(p_K(X)(\omega), \sigma^2)$.

(2.35) Remark: Recall that for $X \in L^2$, $E[X \mid \sigma(K)] = p_{L^2(\Omega, \sigma(K), P)}$ is also an orthogonal projection, but on a larger space. In the Gaussian setting, we may project to a "smaller" G. space.

Proof: (i) Let $Y = X - p_K(X)$. Then Y is orthogonal to X and thus independent of X by (2.22). Hence

$$E[X \mid \sigma(K)] = E[p_K(X) \mid \sigma(K)] + E[Y \mid \sigma(K)] = p_K(X) + E[Y] = p_K(X).$$

(ii) By construction $Y \sim \mathcal{N}(0, \sigma^2)$. Moreover, for $f: \mathbb{R} \to \mathbb{R}_+$ measurable

$$E[f(X) \mid \sigma(K)] = E[f(p_K(X) + Y) \mid \sigma(K)] = \int P_Y(dy)\, f(p_K(X) + y)$$

which implies the claim. (Here we use: if Z is $\mathcal{G}$-measurable, Y independent of $\mathcal{G}$, then $E[g(Y,Z) \mid \mathcal{G}] = \int g(y, Z)\, P_Y(dy)$.)

(2.36) Exercise: Let $X = (X_1, \dots, X_d)$ be a G. vector s.t. $E[X_i] = m_i$, $\mathrm{Cov}(X_i, X_j) = \Sigma_{ij}$. Let $\mathcal{G} = \sigma(X_{k+1}, \dots, X_d)$ for some $k \in \{1, \dots, d\}$.
Then $P[(X_1, \dots, X_k) \in \Gamma \mid \mathcal{G}] = Q(\Gamma, \omega)$, $\Gamma \in \mathcal{B}(\mathbb{R}^k)$
and $Q = \mathcal{N}(\tilde{m}, \tilde{\Sigma})$ with $\tilde{m}_A = m_A + \Sigma_{AB} \Sigma_{BB}^{-1} (X_B - m_B)$
and $\tilde{\Sigma} = \Sigma_{AA} - \Sigma_{AB} \Sigma_{BB}^{-1} \Sigma_{BA}$, where $m = (\underbrace{m_1, \dots, m_k}_{m_A}, \underbrace{m_{k+1}, \dots, m_d}_{m_B})$
and

$$\Sigma = \left(\begin{array}{c|c} \Sigma_{AA} & \Sigma_{AB} \\ \hline \Sigma_{BA} & \Sigma_{BB} \end{array}\right) \begin{array}{l} \updownarrow k \text{ lines} \\ \updownarrow d-k \text{ lines} \end{array}$$

2.4 Gaussian measures

We have seen that checking a function Γ is positive definite quadratic form is hard, c.f. (2.18), (2.19). But, in the situation when T is a vector space with a inner product, $\Gamma(s,t) = \langle s,t \rangle$ is pos. definite, c.f. Gram matrix.

(2.37) <u>Definition:</u> Let $(E, \mathcal{E}, \mu)$ be a measure space, μ-σ-finite. <u>Gaussian measure</u> with intensity μ is a isometry G of $L^2(E,\mathcal{E},\mu)$ with a Gaussian space.

That is for $f, g \in L^2(E,\mathcal{E},\mu)$, $G(f)$ is a centred Gaussian r.v. with $E[G(f)^2] = \|G(f)\|^2_{L^2(P)} = \|f\|^2_{L^2(E,\mathcal{E},\mu)}$
and $E[G(f)G(g)] = \langle f,g \rangle_{L^2(E,\mathcal{E},\mu)}$
If $A \in \mathcal{E}$, $\mu(A) < \infty$, then $G(A) := G(1\!\!1_A) \sim \mathcal{N}(0, \mu(A))$. Further if $A_1, \dots, A_k \in \mathcal{E}$ are disjoint with $\mu(A_i) < \infty$, then the vector $(G(A_1), \dots, G(A_k))$ has diagonal covariance matrix since for $i \neq j$ $E[G(A_i), G(A_j)] = \langle 1\!\!1_{A_i}, 1\!\!1_{A_j} \rangle_{L^2(\mu)} = 0$. Hence, by (2.10), $G(A_i)$'s are independent. Let $A = \bigcup_{i=1}^{\infty} A_i$ for A_i disjoint, i.e. $1\!\!1_A = \sum_{i=1}^{\infty} 1\!\!1_{A_i}$, $\mu(A) < \infty$. By isometry then $G(A) \overset{L^2(P)}{=} \sum_{i=1}^{\infty} G(A_i)$. (and also P-a.s).

Hence, properties of $A \mapsto G(A)$ are similar to the properties of a signed measure (depended on ω). In general, however $A \mapsto G(A)(\omega)$ is not a measure for a fixed $\omega \in \Omega$.

(2.38) <u>Proposition</u> (Existence) Let $(E,\mathcal{E},\mu)$ be measure space, μ-σ-finite. Then there is G. measure with intensity μ.

<u>Proof:</u> Let $(f_i)_{i \in I}$ be a total orthonormal system in $L^2(E,\mathcal{E},\mu)$. Hence every $f \in L^2(\mu)$ can be written as
$$f = \sum_{i \in I} \alpha_i f_i \quad \text{with } \alpha_i = \langle f, f_i \rangle \text{ and } \sum_{i \in I} \alpha_i^2 = \|f\|_2^2 < \infty.$$

We now consider a probability space $(\Omega, \mathcal{A}, P)$ with a iid collection $(X_i)_{i\in I}$ of standard normal r.v.'s. I.e. also $X_i \perp X_j$ in $L^2(P)$, $i\neq j$. Set $G(f) = \sum_{i\in I} \alpha_i X_i$. This series then converge in $L^2(P)$, i.e G is an element of the G. space generated by $(X_i)_{i\in I}$. The isometry property is obvious. □.

(2.39) Remark. When $L^2(E,\mathcal{E},\mu)$ is separable (e.g. when $E=\mathbb{R}$, μ=Lebesgue) the total orthonormal system (f_i) of the last proof is actually a countable ON basis of L^2.

(2.40) Proposition ("quadratic variation" of G. measure)

Let G be a G. measure on $(E,\mathcal{E},\mu)$ and $A\in\mathcal{E}$ with $\mu(A)<\infty$. Assume that there is a sequence $(A_1,\dots,A_{k_n})$, $n\geq 1$ of partitions of A, i.e. $A = A_1^n \cup \dots \cup A_{k_n}^n$, with "step" tending to 0, i.e.

$$\lim_{n\to\infty}\Big(\sup_{1\leq j\leq k_n} \mu(A_j^n)\Big) = 0.$$

Then

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} G(A_j^n)^2 = \mu(A) \quad \text{in } L^2(P)$$

Proof: For n fixed, r.v's $G(A_j^n)$, $1\leq j\leq k_n$, are independent, and $E[G(A_j^n)^2] = \mu(A_j^n)$. Hence

$$E\Big[\Big(\sum_{j=1}^{k_n} G(A_j^n)^2 - \mu(A)\Big)^2\Big] = \sum_{j=1}^{k_n} Var\big(G(A_j^n)^2\big) = 2\sum_{j=1}^{k_n}\mu(A_j^n)^2,$$

since, for $X\sim\mathcal{N}(0,\sigma^2)$, $Var(X^2) = E[X^4]-\sigma^4 \overset{(2.1)}{=} 3\sigma^4-\sigma^4 = 2\sigma^4$.

Finally,

$$2\sum_{j=1}^{k_n}\mu(A_j^n)^2 \leq 2\underbrace{\sup_{1\leq j\leq k_n}\mu(A_j^n)}_{\to 0}\ \underbrace{\sum_{j=1}^{k_n}\mu(A_j^n)}_{=\mu(A)<\infty} \xrightarrow{n\to\infty} 0,$$

proving the L^2-convergence □.

Chapter III. BROWNIAN MOTION

3.1 Pre-Brownian Motion.

(3.1) Definition: Let G be a G. measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ whose intensity is the Lebesgue measure. The process $(B_t)_{t\in\mathbb{R}_+}$ given by

$$B_t = G(\mathbb{1}_{[0,t]})$$

is called pre-Brownian motion

(3.2) Proposition: $(B_t)_{t\geq 0}$ is a Gaussian process with covariance

$$K(s,t) = E[B_s B_t] = s\wedge t.$$

Proof. By definition, B_t, $t\in\mathbb{R}_+$, are elements of the same G. space. That is $(B_t)_{t\geq 0}$ is a G. process. Moreover,

$$E[B_s B_t] = E[G(\mathbb{1}_{[0,s]})\, G(\mathbb{1}_{[0,t]})] = \int_{\mathbb{R}_+} \mathbb{1}_{[0,s]}(r)\,\mathbb{1}_{[0,t]}(r)\, dr = s\wedge t.$$ □

We now give several characterisations of pre-BM.

(3.3) Proposition: A process $(X_t)_{t\geq 0}$ is a pre-BM iff.

(i) $X_0 = 0$ a.s.

(ii) For every $0\leq s<t$, r.v. $X_t - X_s$ is independent of $\sigma(X_r, r\leq s)$ and has $\mathcal{N}(0,t-s)$-distribution.

Proof: $\Rightarrow$. If X_t is pre-BM, then $X_0 = G(\mathbb{1}_{[0,0]})$, so $X_0 = 0$ a.s. To show (ii), let H_s be the G. space generated by X_r, $r\leq s$, and $\tilde{H}_s$ the G. space generated by $(X_{s+u} - X_s;\, u\geq 0)$. Then H_s and $\tilde{H}_s$ are orthogonal, since $E[X_r(X_{s+u}-X_s)] = E[G(\mathbb{1}_{[0,r]})G(\mathbb{1}_{[s,s+u]})] = 0$. Moreover, $H_s, \tilde{H}_s$ are parts of the same G. space, so they are independent by (2.22). In particular $X_t - X_s$ is independent of $\sigma(H_s) = \sigma(X_r, r\leq s)$. Finally, $X_t - X_s = G(\mathbb{1}_{(s,t]}) \sim \mathcal{N}(0,t-s)$.

$\Leftarrow$ Let $0 = t_0 < t_1 < \dots < t_n$. By (ii), $X_{t_i} - X_{t_{i-1}}$, $i\leq n$, are independent and $X_{t_i} - X_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})$. It follows that

X is a G. process. Further, for $f = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{(t_{i-1}, t_i]}$, $\lambda_i \in \mathbb{R}$, we set
$G(f) = \sum_{i=1}^{n} \lambda_i (X_{t_i} - X_{t_{i-1}})$
If g is another such step-function we see from (ii) that
$E[G(f)G(g)] = \int_{\mathbb{R}_+} f(u) g(u) du$ Using the density of step-functions in $L^2(\mathbb{R}_+, dx)$, we see that $f \mapsto G(f)$ can be extended to a isometry of $L^2(\mathbb{R}_+, dx)$ and the G. space gen by X. Moreover, $G([0,t]) = X_t - X_0 = X_t$ by (i), i.e. X is a pre-BM. □

(3.4) <u>Proposition:</u> A process $(X_t)_{t\geq 0}$ is a pre-BM iff $X_0 = 0$ a.s. and for every choice $0 = t_0 < t_1 < .. < t_n$, the random variables $X_{t_i} - X_{t_{i-1}}$, $i \leq n$, are independent and $X_{t_i} - X_{t_{i-1}} \sim N(0, t_i - t_{i-1})$.
In particular, vector $(X_{t_1}, \dots, X_{t_n})$ has density
$$\frac{1}{\sqrt{(2\pi)^n t_1 (t_2 - t_1) \cdots (t_n - t_{n-1})}} \exp\left(- \sum_{i=1}^{n} \frac{(y_i - y_{i-1})^2}{2(t_i - t_{i-1})} \right) \qquad (y_0 = 0)$$

(3.5) <u>Exercise</u>: prove this proposition.

3.2 Invariances of pre-BM

(3.6) <u>Proposition:</u> Let $(B_t)_{t\geq 0}$ be a pre-BM. Then
(a) $-B$ is pre-BM
(b) For every $\lambda > 0$ $B_t^\lambda = \frac{1}{\lambda} B_{\lambda^2 t}$ is a pre-BM (scaling invariance)
(c) $b_t := t B_{1/t}$, $t > 0$, $b_0 := 0$ is a pre BM (time-inversion invariance)
(d) For every $s \geq 0$, $B_t^{(s)} = B_{t+s} - B_s$, $t \geq 0$ is a pre-BM (shift invariance "simple Markov-prop.")

(3.7) <u>Exercise:</u> Prove the converse to (3.2), i.e. If $(X_t)_{t\geq 0}$ is a G. process with covariance $E[X_s X_t] = s \wedge t$, then X is a pre-BM.
Show then the proposition.

(3.8) <u>Notation:</u> Let B be a pre-BM and G the associated G. measure.
For $f \in L^2(\mathbb{R}_+, dx)$ one can write
$$G(f) := \int_0^\infty f(s)\,dB_s \sim \mathcal{N}(0, \|f\|_2^2); \; G(f \mathbb{1}_{[0,t]}) = \int_0^t f(s)\,dB_s .$$
This is justified by
$$\int_u^v dB_s = G((u,v]) = B_v - B_u , \quad 0 \le u < v < \infty .$$
The map $f \mapsto \int_0^\infty f(s)\,dB_s$ is called Wiener integral wrt pre-BM B. Since G. measures are not true measures for a given $\omega \in \Omega$, the Wiener integral is not a true integral. We will, in this lecture, extend the definition to f which may depend on ω.

3.3. <u>Continuity of trajectories</u>

Let $(B_t)_{t \ge 0}$ be a pre-BM. The map $t \mapsto B_t(\omega)$ for a fixed $\omega \in \Omega$ is called the trajectory of the pre-BM B. At this point we cannot say almost anything about it. It is even not clear (or even not true in general) that $t \mapsto B_t(\omega)$ is measurable (say for $\mathbb{P}$-a.e. ω). We are now going to modify B so that its trajectories are continuous.

(3.9) <u>Definition</u>. Let $(X_t)_{t \in T}$, $(Y_t)_{t \in T}$ be two stoch. processes indexed by the same T. Y is called <u>modification</u> of X if
$$\mathbb{P}[X_t = Y_t] = 1 \qquad \forall t \in T.$$

(3.10) <u>Remark</u>: If Y is a modification of X, then they have the same law on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^{\otimes T})$, since $\mathcal{B}(\mathbb{R})^{\otimes T}$ is generated by finite dimensional cylinder events and vectors $(X_{t_1}, \dots, X_{t_n})$, $(Y_{t_1}, \dots, Y_{t_n})$ have the same law for every $\{t_1, \dots, t_n\} \subset T$.
In particular, if X is a pre-BM, then Y is a pre-BM.

(3.11) <u>Remark:</u> X and Y may be rather different. Eg. X may be a.s. continuous and Y a.s discontinuous everywhere.
<u>Exercise:</u> Find an example with $T = \mathbb{R}_+$.

(3.12) <u>Definition:</u> X and Y are called <u>indistinguishable</u> if
$$P(\underbrace{X_s = Y_s \ \forall s \in T}_{\text{should be measurable}}) = 1.$$
Obviously, if X and Y are indistinguishable, then Y is a modification of X. The other implication is false.

(3.13) <u>Lemma:</u> If T is separable and X, Y have a.s. continuous trajectories, then X, Y are indist. $\Leftrightarrow$ X is modif. of Y.

<u>Proof:</u> $\Rightarrow$ obvious
$\Leftarrow$. Let D be a countable dense set in T. Then
$P[X_s = Y_s \ \forall s \in D] = 1$ (countable union) and thus
$P[X_s = Y_s \ \forall s \in T] = 1$ by continuity . $\square$

We now introduce the important result which ensures the existence of continuous modifications

(3.14) Kolmogorov's continuity theorem

<u>Theorem:</u> Let $X = (X_t)_{t\in I}$ be a st. process indexed by a bounded interval $I \subset \mathbb{R}$ taking values in a complete metric space (S, d). Assume that for some $q, \varepsilon, C > 0$

$$E\left[d(X_s, X_t)^q\right] \le C\,|t-s|^{1+\varepsilon} \qquad \forall t, s \in I$$

Then there is a modification $\tilde{X}$ of X which is α-Hölder continuous for every $\alpha \in (0, \frac{\varepsilon}{q})$, that is there is $C_\alpha(\omega)$ s.t.

$$d(\tilde{X}_s, \tilde{X}_t) \le C_\alpha(\omega)\,|t-s|^\alpha \qquad \forall t, s \in I$$

In particular $\tilde{X}$ is an a.s. unique continuous modification of X.

<u>Proof:</u> W.l.o.g we assume $I = [0,1]$, and denote by D the set of all dyadic rationals in $[0,1)$

$$D = \left\{ \sum_{i=1}^{p} \varepsilon_i 2^{-i} : p \in \mathbb{N},\ \varepsilon_i \in \{0,1\} \right\}.$$

Observe also, that it is sufficient to show the result for a $\alpha \in (0, \frac{\varepsilon}{q})$ fixed. Indeed, we may then take a sequence $\alpha_k \uparrow \frac{\varepsilon}{q}$ and observe that the modifications obtained for those α_k are indistinguishable by (3.13).

<u>Step 1:</u> X is α-Hölder on D, i.e. there is $C_\alpha(\omega) < \infty$ s.t.

$$d(X_s, X_t) \le C_\alpha(\omega)\,|t-s|^\alpha \qquad \forall t, s \in D.$$

To see that, observe that the assumption of the theorem implies for $a > 0$, $s, t \in I$

$$P\left[d(X_s, X_t) \ge a\right] \le a^{-q} E\left[d(X_s, X_t)^q\right] \le C a^{-q} |t-s|^{1+\varepsilon}.$$

Taking $s = (i-1)2^{-n}$, $t = i2^{-n}$, $i \in \{1, \dots, 2^n\}$ and $a = 2^{-n\alpha}$

$$P\left[d(X_{(i-1)2^{-n}}, X_{i2^{-n}}) \ge 2^{-n\alpha}\right] \le C\, 2^{nq\alpha}\, 2^{-(1+\varepsilon)n}$$

Hence

$$P\left[\bigcup_{i=1}^{2^n} d(X_{(i-1)2^{-n}}, X_{i2^{-n}}) \ge 2^{-n\alpha}\right] \le C 2^n 2^{nq\alpha} 2^{-(1+\varepsilon)n} = C 2^{-n(\varepsilon - q\alpha)}.$$

For $\alpha \in (0, \frac{\varepsilon}{q})$, this is summable in $n \ge 0$, so by BC lemma P-a.s. $\exists n_0(\omega): n \ge n_0(\omega)\ \forall i \in \{1, \dots, 2^n\}$ $d(X_{(i-1)2^{-n}}, X_{i2^{-n}}) \le 2^{-n\alpha}$, that is

$$K_\alpha(\omega) = \sup_{n \ge 1} \left(\sup_{1 \le i \le 2^n} \frac{d(X_{(i-1)2^{-n}}, X_{i2^{-n}})}{2^{-n\alpha}} \right) < \infty \qquad P\text{-a.s.}$$

We now claim that Step 1 holds with $C_\alpha(\omega) = 2(1-2^{-\alpha})^{-1} K_\alpha(\omega)$.
To this end take $s,t \in D$, $s<t$. Let p be the smallest integer so that $2^{-p} < t-s$. Then for some $k, l, m \in \mathbb{N}$

$$s = k2^{-p} - \varepsilon_{p+1} 2^{-p-1} - \dots - \varepsilon_{p+l} 2^{-p-l}$$
$$t = k2^{-p} + \varepsilon'_p 2^{-p} + \dots + \varepsilon'_{p+m} 2^{-p-m} \quad , \quad \varepsilon_i, \varepsilon'_j \in \{0,1\}.$$

Let
$$s_i = k2^{-p} - \varepsilon_{p+1} 2^{-p-1} - \dots - \varepsilon_{p+i} 2^{-p-i} \qquad 0 \le i \le l$$
$$t_j = k2^{-p} + \varepsilon'_p 2^{-p} + \dots + \varepsilon_{p+j} 2^{-p-j} \qquad 0 \le j \le m.$$

Then, by triangle inequality, P-a.s.

$$\begin{aligned} d(X_s, X_t) = d(X_{s_l}, X_{t_m}) &\le d(X_{s_0}, X_{t_0}) + \sum_{i=1}^{l} d(X_{s_{i-1}}, X_{s_i}) + \sum_{j=1}^{m} d(X_{t_{j-1}}, X_{t_j}) \\ &\le K_\alpha(\omega) \left(2^{-p\alpha} + \sum_{i=1}^{l} 2^{-(p+i)\alpha} + \sum_{j=1}^{m} 2^{-(p+j)\alpha} \right) \\ &\le K_\alpha(\omega)\, 2 \cdot 2^{-p\alpha} (1-2^{-\alpha})^{-1} \le C_\alpha(\omega) (t-s)^\alpha \end{aligned}$$

Step 2. We complete the proof of the theorem. By step 1, $t \mapsto X_t(\omega)$ is a.s. α-Hölder on D, so it is uniformly continuous on D. Since (S,d) is complete, there is a.s. a unique continuous extension of this trajectory to I, namely

$$\tilde{X}_t(\omega) = \lim_{s \to t,\, s \in D} X_s(\omega) \quad , \quad t \in [0,1] \text{ on } \{K_\alpha < \infty\},$$

and this extension is also α-Hölder. If $K_\alpha = \infty$, then we set $X_t = x_0$ for all $t \ge 0$ and some fixed $x_0 \in S$.

To see that $\tilde{X}$ is a modification of X, observe that by assumption

$$\lim_{s \to t} X_s = X_t \quad \text{in probability} \quad \forall t \in [0,1].$$

By construction $\tilde{X}_t \overset{a.s.}{=} \lim_{s \to t,\, s \in D} X_s$, which implies easily that $X_t = \tilde{X}_t$ a.s.

$\square$

(3.15) Remark: When I is not bounded, e.g. $I = \mathbb{R}_+$, one can apply the theorem on $[0,1], [1,2], \dots$ and one finds a modification that is locally α-Hölder continuous for every $\alpha \in (0, \frac{\varepsilon}{q})$.

An important corollary of Kolmogorov's continuity theorem (3.14) is:

(3.16) <u>Corollary</u>: Let B be a pre-BM. Then B has a modification which is locally α-Hölder (i.e. also continuous) for every $\alpha \in (0, \frac{1}{2})$.

<u>Proof.</u> For $s<t$, $B_t - B_s \sim \mathcal{N}(0, t-s)$, that is $B_t - B_s \overset{law}{=} \sqrt{t-s} \cdot N$, where $N \sim \mathcal{N}(0,1)$. Hence for every $q>0$

$$E[|B_t - B_s|^q] = (t-s)^{q/2} E[|N|^q] = C_q (t-s)^{q/2}.$$

If $q>2$, we may apply the theorem with $\varepsilon = \frac{q}{2}-1$ to find a modification which is α-Hölder for $\alpha < \frac{q-2}{2q}$. Taking $q \uparrow \infty$, we prove the claim.

3.4 <u>Brownian motion</u>

(3.17) <u>Definition</u>: A process $B = (B_t)_{t \geq 0}$ is a Brownian motion if

(i) B is a pre-BM

(ii) The trajectories of B, i.e. the maps $t \mapsto B_t(\omega)$, are continuous for every $\omega \in \Omega$.

(3.18) <u>Remark</u>: Existence of BM follows from (3.16). Indeed, any modification of a pre-BM is again a pre-BM.

(3.19) <u>Remark</u>: Proposition (3.6) remains true if we replace all occurrences of "pre-BM" by "BM". Indeed, the continuity of the trajectories is (almost) immediate, with exception:

(3.20) <u>Exercise</u>: Show that $\beta_t = t B_{1/t}$ is continuous in 0.

(3.21) <u>Remark</u>: In Propositions (3.3), (3.4) one should, in addition, require the continuity of trajectories. I.e. e.g. (3.3) becomes

X is a BM $\Leftrightarrow$ (a) $X_0 = 0$, (b) $t \mapsto X_t(\omega)$ is cont for every ω

(c) for $0<s\leq t$, $X_t - X_s \sim \mathcal{N}(0, t-s)$, $X_t - X_s \perp\!\!\!\perp \sigma(X_r, r \leq s)$.

Wiener measure. Let $C = C(\mathbb{R}_+, \mathbb{R})$ and $\mathcal{F}$ be the smallest σ-field making the canonical coordinates $X_t : C \to \mathbb{R}$, $w \in C \mapsto X_t(w) = w(t)$ measurable. (This σ-field coincides with the Borel-σ-field of the topology of uniform convergence on compacts). Let B be a BM on some probability space $(\Omega, \mathcal{A}, P)$. B induces a measurable map

$$\Omega \longrightarrow C(\mathbb{R}_+, \mathbb{R})$$
$$\omega \longmapsto (t \mapsto B_t(\omega)).$$

The Wiener measure is the image of P by this map, $W := B_* P$, that is W is the law of BM on C.

By (3.4), for $0 = t_0 < t_1 < \ldots < t_m$, $A_0, \ldots, A_m \in \mathcal{B}(\mathbb{R})$,

(3.22) $$W(\{w : w(t_0) \in A_0, \ldots, w(t_m) \in A_m\}) = \mathbb{1}_{A_0}(0) \int_{A_1 \times \ldots \times A_m} \frac{dy_1 \ldots dy_m}{(2\pi)^{m/2} \sqrt{t_1(t_2 - t_1) \cdots (t_m - t_{m-1})}} \exp\left\{ - \sum_{i=1}^{m} \frac{(y_i - y_{i-1})^2}{2(t_i - t_{i-1})} \right\}$$

We know that finite-dimensional distributions characterize measures on $(C, \mathcal{C})$, that is (3.22) defines W uniquely. In particular the Wiener measure does not depend on which BM we used for its construction.

If one take the probability space

$$\Omega = C, \quad \mathcal{A} = \mathcal{F}, \quad P = W$$

then the canonical process $X_t(w) = w(t)$ is a BM. This is the canonical construction of BM.

d-dim. Wiener measure. Similar construction works with $C = C(\mathbb{R}_+, \mathbb{R}^d)$ and can. coordinates $X_t : C \to \mathbb{R}^d$. Taking $B^{(1)}, \ldots, B^{(d)}$ be d indep. copies of BM on $(\Omega, \mathcal{A}, P)$ and setting $W = B_* P$ for

$$\omega \xmapsto{B} (t \mapsto (B^{(1)}_t, \ldots, B^{(d)}_t)).$$

(3.32) remains valid as it is with $A_i \in \mathcal{B}(\mathbb{R}^d)$.

3.5 Lévy-Ciesielski construction of BM. (*)

One can construct a BM without Donsker's or Kolmogorov's thm, by taking a particular suitable basis in $L^2(\mathbb{R}_+, dx)$ for the construction of the G. measure corresponding to pre-BM:

$$\text{Let } \varphi_\ell = 1\!\!1_{[\ell,\ell+1)}, \quad \ell \in \mathbb{N}$$

$$\left.\varphi_{m,k} = 2^{m/2}\left(1\!\!1_{\left[\frac{k}{2^m}, \frac{k+\frac12}{2^m}\right)} - 1\!\!1_{\left[\frac{k+\frac12}{2^m}, \frac{k+1}{2^m}\right)}\right) \quad m,k \in \mathbb{N} \right\} \text{Haar functions}$$

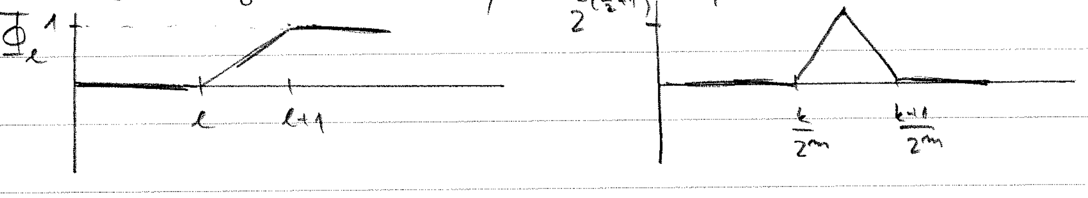
be a complete ON basis of $L^2(\mathbb{R}_+, dx)$. For $f \in L^2(\mathbb{R}_+, dx)$, let

$$G(f) = \sum_\ell \xi_\ell \langle f, \varphi_\ell \rangle + \sum_{m,k} \xi_{m,k} \langle f, \varphi_{m,k} \rangle$$

where $\xi_\ell, \xi_{m,k}$ are iid $\mathcal{N}(0,1)$. G is a G. measure and

$$B_t = G(1\!\!1[0,t]) = \sum_\ell \xi_\ell \Phi_\ell(t) + \sum_{m,k} \xi_{m,k} \Phi_{m,k}(t), \text{ where}$$

$$\Phi_\ell(t) = \int_0^t \varphi_\ell(s)\,ds \quad ; \quad \Phi_{m,k}(t) = \int_0^t \varphi_{m,k}(s)\,ds.$$



B_t is a pre-BM by definition. Moreover, if $n_0 \in \mathbb{N}$, $t \le n_0$

$$B_t^n = \sum_{\ell \le n} \xi_\ell \Phi_\ell(t) + \sum_{m \le n} \sum_{k \le n_0 2^m} \xi_{m,k} \Phi_{m,k}(t),$$

then one can show that B_t^n converges uniformly on $[0, n_0]$ to B_t. Since B_t^n are continuous in t, this yields another proof of existence of BM.

3.6 (Quadratic) variation of BM trajectories.

(3.23) <u>Proposition:</u> Let $0 = t_0^n < \quad < t_{k_n}^n = t$ be a sequence of sub-divisions of $[0,t]$ with step-size tending to 0, $\sup_i t_i^n - t_{i-1}^n \xrightarrow{n\to\infty} 0$.
Then
$$\lim_{n\to\infty} \sum_{i=1}^{k_n} \left(B_{t_i^n} - B_{t_{i-1}^n}\right)^2 = t \quad \text{in } L^2.$$

<u>Proof:</u> Recall that $B_{t_i^n} - B_{t_{i-1}^n} = G\left((t_{i-1}^n, t_i^n]\right)$. The claim is then consequence of Proposition (2.40) $\square$

(3.24) <u>Corollary</u> The trajectories $t \mapsto B_t$ have P-a.s. infinite variation on every non-trivial interval

<u>Proof:</u> W.l.o.g. we consider intervals $[0,t]$, $t \in \mathbb{Q}$ only. Recall that variation of $w \in C$ over $[0,t]$ is given by
$$V_t(w) = \sup_{\substack{0=t_0<\dots<t_n\le t \\ \text{rationals}}} \sum_{i\le n} |w(t_i) - w(t_{i-1})|$$

Assume that $P[V_t(B) < \infty] > 0$. Then
$$\sum_{i=1}^{k_n} (B_{t_i^n} - B_{t_{i-1}^n})^2 \le \underbrace{\sup_{i\le k_n} |B_{t_i^n} - B_{t_{i-1}^n}|}_{\xrightarrow{n\to\infty} 0} \underbrace{\sum_{i=1}^{k_n} |B_{t_i^n} - B_{t_{i-1}^n}|}_{\le V_t(B)}$$
$$\xrightarrow{n\to\infty} 0 \text{ on } \{V_t(B) < \infty\}.$$

This contradicts (3.23) $\square$

3.7 Simple Markov property of Wiener measure

Let $C = C(\mathbb{R}_+, \mathbb{R}^d)$, $X_t : C \to \mathbb{R}^d$ be the canonical coordinates, $\mathcal{F} = \sigma(X_t : t \geq 0)$, W be the Wiener measure. Let further $x \in \mathbb{R}^d$ and W_x be the law of BM started from x given by

W_x = image of W under the map $C \ni w \mapsto w(\cdot) + x \in C$,

and E_x the corresponding expectation. Then, by (3.22),

$$E_x[h(X_{t_0}, \dots, X_{t_n})] = E^W[h(X_{t_0}+x, \dots, X_{t_n}+x)] =$$
$$= \int_{\mathbb{R}^n} h(x, y_1, \dots, y_n) \prod_{i=1}^{n} \frac{dy_i}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left\{-\frac{(y_i - y_{i-1})^2}{2(t_i - t_{i-1})}\right\}$$

for $0 = t_0 < \dots < t_n$, $h \in b\mathcal{B}(\mathbb{R}^{(n+1)d})$, with $y_0 = x$. Let Θ_s, $s \geq 0$, be the shifts on C, $\Theta_s(w)(\cdot) = w(s+\cdot) \in C$. Finally, let

$$\mathcal{F}_t = \sigma(X_s : s \leq t)$$
$$\mathcal{F}_t^+ = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$$

$\mathcal{F}_t$ describes the past before t and $\mathcal{F}_t^+$ peaks infinitesimally into the future.

(3.25) Example: Consider the event "the trajectory leaves immediately its starting point" $= \bigcap_{n \geq 1} \left(\bigcup_{r \in \mathbb{Q} \cap [0, \frac{1}{n}]} \{X_r \neq X_0\} \right)$

This event is in $\mathcal{F}_0^+$ but not in $\mathcal{F}_0$

We now extend (3.6(d)).

(3.26) Theorem (simple Markov Property)

Let $Y \in b\mathcal{F}$, $s \geq 0$, $x \in \mathbb{R}^d$. Then

(i) $E_x[Y \circ \Theta_s \mid \mathcal{F}_s^+] = E_{X_s}[Y]$ $\quad W_x$-a.s.

(ii) Under W_x, $(X_{s+u} - X_s)_{u \geq 0}$ is a BM independent of $\mathcal{F}_s^+$.

Proof. Step 1 (exercise) $y \mapsto E_y[Y]$ is Borel measurable for all $Y \in b\mathcal{F}$

(this is necessary, otherwise the RHS of (i) is not a r.v.)

Step 2 For every $A \in \mathcal{F}_s$ and $g \in b\mathcal{B}(\mathbb{R}^{(n+1)d})$, $s \geq 0$, $0 = t_0 < \dots < t_n$

(3.27) $$E_x\left[1\!\!1_A\, g(X_{s+t_0}, \dots, X_{s+t_n})\right] = E_x\left[1\!\!1_A\, E_{X_s}\left[g(X_{t_0}, \dots, X_{t_n})\right]\right]$$

This follows from (3.6(d)) and definition of W_x.

Step 3: For every $A \in \mathcal{F}_s^+$, $Y \in b\mathcal{F}$, $s \geq 0$

(3.28) $$E_x\left[1\!\!1_A\, Y \circ \Theta_s\right] = E_x\left[1\!\!1_A\, E_{X_s}[Y]\right]$$

Indeed, let $g \in bC(\mathbb{R}^{(n+1)d})$. Then, by def of W_x and DCT

$$y \in \mathbb{R} \longmapsto E_y\left[g(X_{t_0}, \dots, X_{t_n})\right] \in bC(\mathbb{R})$$

Using then step 2 with $s+\varepsilon$ on the place of s, $A \in \mathcal{F}_s^+ \subset \mathcal{F}_{s+\varepsilon}$, let $\varepsilon \downarrow 0$ and using DCT again, we see that (3.27) holds also for $g \in bC(\mathbb{R}^{d(n+1)})$, $s \geq 0$, $A \in \mathcal{F}_s^+$. (Here one also uses that BM has continuous trajectories !, i.e. $X_{s+\varepsilon} \xrightarrow{\varepsilon \downarrow 0} X_s$)

By approximation, (3.27) then holds for $g(x_0, \dots, x_n) = \prod_{i=0}^{n} 1\!\!1_{F_i}(x_i)$, $F_i \subset \mathbb{R}^d$ closed, and thus, by Dynkin's lemma, (3.28) holds for $Y = 1\!\!1_{A'}$, $A' \in \mathcal{F}$. Another approximation step yields (3.28) for all $Y \in b\mathcal{F}$.

The claim (i) of (3.26) follows directly from Step 3.

Step 4: Proof of (ii). $(X_{u+s} - X_u)_{u \geq 0}$ has continuous trajectories. Moreover,

$$E_x\left[f(X_{s+t_0} - X_s, \dots, X_{s+t_n} - X_s) \mid \mathcal{F}_s^+\right] = E_x\left[f(X_{t_0} - X_0, \dots, X_{t_n} - X_0) \circ \Theta_s \mid \mathcal{F}_s^+\right]$$
$$\overset{(i)}{=} E_{X_s}\left[f(X_{t_0} - X_0, \dots, X_{t_n} + X_0)\right] = E_0\left[f(X_{t_0}, \dots, X_{t_n})\right],$$

proving (b) $\square$.

As corollary we obtain

(3.29) **Theorem** (Blumenthal's 0-1 law) For $x \in \mathbb{R}$, $A \in \mathcal{F}_0^+$,

$$W_x(A) \in \{0, 1\}.$$

Proof: This follows from

$$\begin{aligned} 1\!\!1_A &= E_x\left[1\!\!1_A \mid \mathcal{F}_0^+\right] && (A \in \mathcal{F}_0^+) \\ &= E_x\left[1\!\!1_A \circ \Theta_0 \mid \mathcal{F}_0^+\right] && (\Theta_0 = \mathrm{Id}) \\ &\overset{as}{=} E_{X_0}\left[1\!\!1_A\right] && (\text{Markov property}) \\ &= E_x\left[1\!\!1_A\right] = W_x(A). \end{aligned}$$

$\square$

(3.30) Examples

(i) Let $\tilde{H}_\pm := \inf\{s>0 : X_s \gtrless 0\}$ be the hitting times of $(0,\infty)$ or $(-\infty,0)$.

Then W_0-a.s. $\tilde{H}_+ = \tilde{H}_- = 0$.

Proof. $\{H_+ = 0\} = \bigcap_{n\geq 1} \Big(\bigcup_{t\in[0,\frac{1}{n}]\cap\mathbb{Q}} \{X_t > 0\}\Big) \in \mathcal{F}_0^+$.

Further, for every $t>0$, $W_0(X_t>0) = \frac{1}{2}$. So,

$W_0(\tilde{H}_+ \leq t) \geq W_0(X_t > 0) = \frac{1}{2}$. Hence

$W_0(\tilde{H}_+ = 0) = \lim\downarrow W_0(H_+ \leq t) \geq \frac{1}{2}$, i.e. $W_0(H_+ = 0) = 1$ by (3.29)

(ii) $\forall \varepsilon>0$ W_0-a.s. $\sup\limits_{0\leq s\leq\varepsilon} X_s > 0$, $\inf\limits_{0\leq s\leq \varepsilon} X_s < 0$.

Proof. Let $A = \bigcap_{n\geq 1} \{\sup\limits_{s\leq \frac{1}{n}} X_s > 0\} \in \mathcal{F}_0^+$. By the same arguments as before $W_0(A) \geq \frac{1}{2}$, so $W_0(A) = 1$ by (3.29). Since

$\{\sup\limits_{s\leq\varepsilon} X_s > 0\} \supset \{\sup\limits_{s\leq\frac{1}{n}} X_s > 0\}$ for some n, the claim follows.

(iii) Let $H_a = \inf\{t\geq 0 : X_t = a\}$, $a\in\mathbb{R}$. Then W_0-a.s. $T_a < \infty$.

and thus $\limsup\limits_{t\to\infty} X_t = -\liminf\limits_{t\to\infty} X_t = +\infty$.

Proof: We have

$1 = P[\sup\limits_{s\leq 1} X_s > 0] = \lim\limits_{\delta\downarrow 0}\uparrow P[\sup\limits_{s\leq 1} X_s > \delta]$

by scaling invariance (3.<(b)) $\lambda = 1/\delta$

$= \lim\limits_{\delta\downarrow 0}\uparrow P[\sup\limits_{s\leq \frac{1}{\delta^2}} X_s > 1] = P[\sup\limits_{s\geq 0} X_s > 1] = P[H_1 < \infty]$

Using the scaling invariance again, this holds for $a>0$, and by reflection invariance for $a\in\mathbb{R}$. The second claim follows by continuity $\square$

Exercise: (a) $d\geq 2$. Let K be an open cone in $\mathbb{R}^d$ and $(X_s)_{s\geq 0}$ a d-dim BM. Let $\tilde{H}_K = \inf\{s>0 : X_s \in K\}$. Then W_0-a.s. $\tilde{H}_K = 0$

(b) $d=1$, $t_n \downarrow 0$, $t_n > 0$. Show that

$$\limsup_{n\to\infty} X(t_n)/\sqrt{t_n} = +\infty \quad W_0\text{-a.s.}$$

3.8. Filtrations & stopping times.

Recall that for a discrete filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_n), P)$ (i.e. for $(\Omega, \mathcal{A}, P)$ with a sequence of σ-algebras $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \dots \subset \mathcal{A}$), a stopping time T is a $\mathbb{N} \cup \{\infty\}$-valued r.v. such that $\{T = n\} \in \mathcal{A}_n$ for every $n \in \mathbb{N}$. I.e. to decide "whether we stop at time n", we should know "only the past up to n".

This notion naturally generalises to continuous time.

(3.31) Definition: Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge0})$ be a filtered probability space, i.e. $(\mathcal{G}_t)_{t\ge0}$ is a filtration = a sequence of σ-algebras, $\mathcal{G}_s \subseteq \mathcal{G}_t \subseteq \mathcal{G}$, $0 \le s \le t$. A map $T: \Omega \to [0, \infty]$ is a $(\mathcal{G}_t)$-stopping time if

$$\{T \le t\} \in \mathcal{G}_t \quad \text{for all } t \ge 0.$$

The σ-algebra of the "past of T" is given by

$$\mathcal{G}_T = \{A \in \mathcal{G} : A \cap \{T \le n\} \in \mathcal{G}_t \text{ for all } t \ge 0\}.$$

(3.32) Exercise: Show that $\mathcal{G}_T$ is a σ-algebra.

(3.33) Example: (Entrance time of a closed set)

Consider the space $C = C(\mathbb{R}_+, \mathbb{R}^d)$ with can. coordinates $X_t : C \to \mathbb{R}^d$ and canonical filtration $\mathcal{F}_t = \sigma(X_s, s \le t)$, $\mathcal{F} = \sigma(X_s, s \ge 0)$. Let $A \subset \mathbb{R}^d$ be a closed set. The entrance time of X into A is

$$H_A = \inf\{s \ge 0 : X_s \in A\}. \qquad (H_A = \infty \text{ if } \{\dots\} = \emptyset).$$

Then H_A is a stopping time. Indeed, since A is closed and $w \in C$ are continuous, $\{s \ge 0 : X_s(w) \in A\}$ is a closed subset of $[0, \infty)$ containing $H_A(w)$ (when $H_A(w) < \infty$). Hence $H_A > t \iff \forall s \le t \ \operatorname{dist}(X_s(w), A) > 0 \iff \inf_{s \in [0,t]} \operatorname{dist}(X_s(w), A) > 0$. Hence

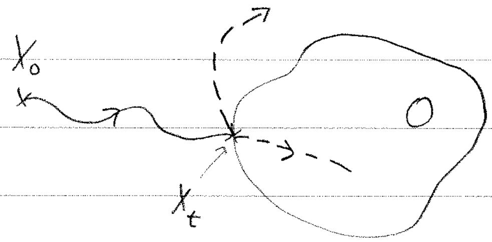
$$\{H_A > t\} = \bigcup_{n \ge 1} \bigcap_{s \in [0,t] \cap \mathbb{Q}} \{\operatorname{dist}(X_s(w), A) > \tfrac{1}{n}\} \in \mathcal{F}_t,$$

proving the claim.

(3.34) <u>Example</u> (Entrance time of a open set)

In the setting of the last example, replace A by an open set $O \subset \mathbb{R}^d$.

$H_O = \inf\{s \geq 0 : X_s \in O\}$. Observe that $\{H_O = t\} \notin \mathcal{F}_t$.



two trajectories agreeing up to time t with $H_O = t$ for one and $H_O > t$ for another.

Here, the filtration

$$\mathcal{F}_t^+ = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$$

comes to help, since it "peaks into the future". We claim:

(3.35) H_O is $(\mathcal{F}_t^+)$-stopping time. Indeed,

$$\{H_O < s\} = \bigcup_{u \in [0,s) \cap \mathbb{Q}} \{X_u \in O\} \in \mathcal{F}_s \quad \text{for } s \geq 0.$$

and

$$\{H_O \leq s\} = \bigcap_{\varepsilon > 0} \{H_O < s + \varepsilon\} \in \mathcal{F}_s^+$$

(3.36) <u>Definition</u>: A filtration $(\mathcal{G}_t)$ is called <u>right-continuous</u> if $\mathcal{G}_t = \bigcap_{\varepsilon > 0} \mathcal{G}_{t+\varepsilon}$.

(3.37) <u>Exercise</u>: • If $(\mathcal{G}_t)$ is right-continuous, then T is st. time $\Leftrightarrow \{T < t\} \in \mathcal{G}_t \; \forall t$.

• $(\mathcal{F}_t^+)_{t \geq 0}$ is a right-continuous filtration.

(3.38) <u>Proposition</u>: $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0})$

(i) T is $\mathcal{G}_t$-st time $\Rightarrow$ T is $\mathcal{G}_T$-measurable

(ii) S, T - st. times $\Rightarrow$ $S \vee T$, $S \wedge T$ are st. times.

(iii) $(\Omega, \mathcal{G}) = (C, \mathcal{F})$. If T, S are $\mathcal{F}_t^+$-st. times, so is

$$T + S \circ \theta_T = \begin{cases} T(\omega) + S(\theta_{T(\omega)}(\omega)) & \text{if } T(\omega) < \infty \\ \infty & \text{if } T(\omega) = \infty \end{cases}$$

(3.39) <u>Exercise</u>: Prove (i) and (ii)

<u>Proof of (iii)</u>: By (3.37), $(\mathcal{F}_t^+)$ is right continuous, so we need to show that

(3.40) $$\{T + S \circ \theta_T < t\} \in \mathcal{F}_t^+$$

Note that

$$\{T + S \circ \theta_T < t\} = \bigcup_{\substack{u, v \in \mathbb{Q} \cap (0,\infty) \\ u+v<t}} \{T < u, S \circ \theta_T < v\}.$$

We claim that when $\{T<u\} \neq \emptyset$, then

(3.41) $$\theta_T : (\{T<u\}, \mathcal{F}_{u+v} \cap \{T<u\}) \to (C, \hat{\mathcal{F}}_v) \text{ is measurable}$$

Indeed, for $s \in [0,v]$, $X_s \circ \theta_T$ is measurable as a map from $(\{T<u\}, \mathcal{F}_{v+u} \cap \{T<u\})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as follows from

$$X_s \circ \theta_T(w) = X_{s+T(w)}(w) = \lim_{m\to\infty} \sum_{k=1}^{m} X_{s+\frac{k}{m}u}(w)\, 1\!\!1\{T \in [\tfrac{k-1}{m}u, \tfrac{k}{m}u)\};$$

the first summand is $\mathcal{F}_{s+u} \cap \{T<u\}$, hence $\mathcal{F}_{u+v} \cap \{T<u\}$-measurable and the second is $\mathcal{F}_u^+ \cap \{T<u\}$ and thus $\mathcal{F}_{u+v} \cap \{T<u\}$-measurable. Since, $\hat{\mathcal{F}}_v$ is the smallest σ-algebra making X_s, $s \in [0,v]$, measurable, (3.41) follows.

Further, $\{S<v\} = \bigcup_{r<v, r\in\mathbb{Q}} \{S \le r\} \in \hat{\mathcal{F}}_v$, hence

$$\{T<u, S\circ\theta_T < v\} = \{w \in \{T<u\}, \theta_T(w) \in \{S<v\}\}$$
$$\overset{(3.41)}{\in} \mathcal{F}_{u+v} \cap \{T<u\} \subset \mathcal{F}_{u+v}, \text{ since } \{T<u\} \in \mathcal{F}_u^+.$$

Going back, this yields $\{T+S\circ\theta_T<t\} \subset \mathcal{F}_t \subset \mathcal{F}_{t^+}$, proving (3.40) □

(3.42) <u>Exercise</u> $(\Omega, \mathcal{G}, (\mathcal{G}_t))$, T, S - st. times.

(i) $S \le T \Rightarrow \mathcal{G}_S \subseteq \mathcal{G}_T$

(ii) both $\{S<T\}$ and $\{S\le T\}$ belong to $\mathcal{G}_S \cap \mathcal{G}_T$

(iii) $A \in \mathcal{G}_S \Rightarrow \{S<T\}\cap A, \{S\le T\}\cap A \in \mathcal{G}_{S\wedge T}$.

3.9. Strong Markov Property

We now generalise the simple Markov property (3.26) to stopping times. To this end we need a last definition.

(3.43) Definition: When T is a $\mathcal{F}_t^+$-stopping time, we set

$$\mathcal{F}_T^+ = \{A \in \mathcal{F} : A \cap \{T \le t\} \in \mathcal{F}_t^+ \ \forall t \ge 0\}$$
$$= \{A \in \mathcal{F} : A \cap \{T < t\} \in \mathcal{F}_t^+ \ \forall t \ge 0\}$$

(to see the 2^{nd} equality observe that "$\subset$" is trivial and for "$\supset$" write $A \cap \{T \le t\} = \bigcap_{n \ge 1} A \cap \{T < t + \frac{1}{n}\} \in \mathcal{F}_t^+$)

(3.44) Theorem: (strong Markov property of Wiener measure)

T an $\mathcal{F}_t^+$-stopping time, $Y \in b\mathcal{F}$, $x \in \mathbb{R}^d$. Then

(3.45) $$E_x [Y \circ \Theta_T \mid \mathcal{F}_T^+] = E_{X_T}[Y] \qquad W_x\text{-a.s. on } \{T < \infty\}.$$

(3.46) Remark: The claim of the theorem contains implicitly that:

(i) $\Theta_T : (\{T<\infty\}, \mathcal{F} \cap \{T<\infty\}) \to (C, \mathcal{F})$ is measurable

(ii) the r.v. $E_{X_T}[Y]$ defined on $\{T<\infty\}$ is in $\mathcal{F}_T^+ \cap \{T<\infty\}$.

Moreover (3.45) is equivalent with

For every $A \in \mathcal{F}_T^+ \cap \{T<\infty\}$: $E_x[Y \circ \Theta_T \, \mathbb{1}_A] = E_x[\mathbb{1}_A \, E_{X_T}[Y]]$.

We need a preparatory step to show (3.46 (ii))

(3.47) Definition: Given $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0})$, an $\mathbb{R}^d$-valued process $(Z_u)_{u \ge 0}$ is called $\mathcal{G}$-adapted if $Z_t \in \mathcal{G}_t$. It is called progressively measurable if the restriction of $Z_t(\omega)$ to $[0,t] \times \Omega$ is $\mathcal{B}([0,t]) \times \mathcal{G}_t$-measurable for every $t \ge 0$.

(3.48) Example: If Z_t is adapted and right-continuous, then it is progr. measurable.

Indeed, on $[0,t] \times \Omega$,

$$Z_s(\omega) = \lim_{n\to\infty} \sum_{k=1}^{n} Z_{\frac{k}{n}t}(\omega) \, \mathbb{1}\{\tfrac{k-1}{n}t \le s < \tfrac{k}{n}t\} + Z_t(\omega) \, \mathbb{1}\{s = t\}$$

and the summands are obviously $\mathcal{B}([0,t]) \times \mathcal{G}_t$-measurable.

(3.49) <u>Lemma:</u> $(\Omega, \mathcal{G}, (\mathcal{G}_t))$, (Z_u) - progressively measurable, T-$(\mathcal{G}_t)$-stopping time: Then, the map $Z_T: \{T<\infty\} \ni \omega \mapsto Z_{T(\omega)}(\omega) \in \mathbb{R}^d$ is $\mathcal{G}_T \cap \{T<\infty\}$-measurable.

<u>Proof:</u> By definition of the σ-algebra $\mathcal{G}_T$, it is sufficient to show

(3.50) $$\left(\{T\le t\}, \mathcal{G}_t \cap \{T\le t\}\right) \xrightarrow{Z_T} (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ is measurable } \forall t\ge 0.$$

This map is a composition of measurable maps.

$$\omega \in \left(\{T\le t\}, \mathcal{G}_t \cap \{T\le t\}\right) \longmapsto (T(\omega), \omega) \in \left([0,t]\times\Omega, \mathcal{B}([0,t])\otimes \mathcal{G}_t\right)$$

$$(u,\omega) \in \left([0,t]\times\Omega, \mathcal{B}([0,t])\otimes \mathcal{G}_t\right) \longmapsto Z_u(\omega) \in (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)).$$

The first one is measurable because both maps:

$$\left(\{T\le t\}, \mathcal{G}_t \cap \{T\le t\}\right) \xrightarrow{T} \left([0,t], \mathcal{B}([0,t])\right) \text{ and}$$

$$\left(\{T\le t\}, \mathcal{G}_t \cap \{T\le t\}\right) \xrightarrow{Id} (\Omega, \mathcal{G}_t)$$

are measurable. The second one is measurable because Z is progressively measurable $\square$.

<u>Proof of Theorem (3.44).</u>

<u>Step 1:</u> (Remark (3.46)(i)) $\theta_T: (\{T<\infty\}, \mathcal{F}\cap\{T<\infty\}) \to (C, \mathcal{F})$ is measurable

<u>Proof:</u> It is sufficient to show that for any $s\ge 0$,

(3.51) $$X_s \circ \theta_T: (\{T<\infty\}, \mathcal{F}\cap\{T<\infty\}) \longrightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ is}$$

measurable. The claim then follows as in Exercise 2.7.

We write for $w\in\{T<\infty\}$, using that BM is (right-) continuous,

$$X_s\circ\theta_T(w) = X_{s+T(w)}(w) = \lim_{n\to\infty} \sum_{k\ge 1} X_{s+\frac{k}{n}}(w)\, \mathbb{1}\left\{\tfrac{k-1}{n} \le T < \tfrac{k}{n}\right\}.$$

From this the measurability of (3.51) follows $\square$.

<u>Step 2:</u> $X_T: (\{T<\infty\}: \mathcal{F}_T^+ \cap \{T<\infty\}) \longrightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is measurable.

<u>Proof:</u> By Example (3.48), the BM is progressively measurable. The claim then follows from lemma (3.49).

<u>Step 3:</u> (Remark (3.46)(ii)) $(\{T<\infty\}, \mathcal{F}_T^+\cap\{T<\infty\}) \xrightarrow{E_{X_T}[Y]} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable

<u>Proof:</u> We have shown that the map $y\in\mathbb{R}^d \longmapsto E_y[Y]$ is measurable in (3.7). (i.e. $(y\in\mathbb{R}^d, A\in\mathcal{F}) \longmapsto W_y(A)$ is a stoch. kernel)

Combining this with Steps 1,2 the claim follows. $\square$.

Step 4: (3.45) holds true if T takes value in an at most denumerable set of values in $[0,\infty]$.

Proof: Let $(a_n)_{0\le n\le N(\le\infty)}$ be the set of values of T in $[0,\infty)$. For $A\in\mathcal{F}_T^+\cap\{T<\infty\}$, $0=t_0<\dots<t_k$, $f\in b\mathcal{B}(\mathbb{R}^{d(k+1)})$ we have

$$E_x[1_A f(X_{t_0},\dots,X_{t_m})\circ\theta_T]=E_x[1_A f(X_{t_0+T},\dots,X_{t_k+T})]=$$
$$=\sum_n E_x[1_{A\cap\{T=a_n\}} f(X_{t_0+a_n},\dots,X_{t_k+a_n})]$$

Since $A\cap\{T=a_n\}$ is in $\mathcal{F}^+_{a_n}$, by the simple Markov property

$$=\sum_n E_x[E_{X_{a_n}}[f(X_{t_0},\dots,X_{t_m})]1_{\{T=a_n\}\cap A}]$$
$$=E_x[E_{X_T}[f(X_{t_0},\dots,X_{t_m})]1_A].$$

Dynkin's lemma then gives (3.46)(iii) and thus (3.45) $\square$.

Step 5: Proof of (3.45) for a general stopping time T.

We use discrete skeleton approximation of T:

(3.52) $$T_n=\sum_{k\ge0}\frac{k+1}{2^n}1\{\tfrac{k}{2^n}\le T<\tfrac{k+1}{2^n}\}+\infty 1\{T=\infty\}.$$

Observe that T_n is $\mathcal{F}_t^+$-stopping time and $T_n\downarrow T$ as $n\to\infty$. Indeed, the second claim is obvious, For the first one for $k,n\ge0$, $\{T_n\le\frac{k+1}{2^n}\}=\{T<\frac{k+1}{2^n}\}\in\mathcal{F}^+_{\frac{k+1}{2^n}}$ and for $t\in[\frac{k}{2^n},\frac{k+1}{2^n})$ $\{T_n\le t\}=\{T_n\le\frac{k}{2^n}\}\in\mathcal{F}^+_{\frac{k}{2^n}}\subset\mathcal{F}^+_t$, so T_n is $\mathcal{F}_t^+$-stopping time. Moreover, from $T<T_n$ follows $\mathcal{F}_T^+\subset\mathcal{F}^+_{T_n}$.

Take now $A\in\mathcal{F}_T^+\cap\{T<\infty\}$. From $\{T_n=\infty\}=\{T=\infty\}$ we see that $A\in\{T_n<\infty\}\cap\mathcal{F}^+_{T_n}$. Using the step 4, we get for $0=t_0<\dots<t_k$, $f_0,\dots,f_k$ bounded, continuous.

$$E_x[\prod_{i=0}^k f_i(X_{t_i+T_n})1_A]=E_x[E_{X_{T_n}}[\prod_{i=0}^k f_i(X_{t_i})]1_A].$$

Using the facts that $(X_\cdot)$ is (right-) continuous, f_i are continuous, $y\longmapsto E_y[\prod_{i=0}^k f_i(X_{t_i})]$ is continuous (which was shown in (3.10)) and the dominated convergence, we get by $n\to\infty$,

$$E_x[\prod_{i=0}^k f_i(X_{T+t_i})1_A]=E_x[E_{X_T}[\prod f_i(X_{t_i})]1_A].$$

This claim can be then extended to an arbitrary $Y\in b\mathcal{F}$ using the same argument as around (3.11). $\square$

3.10 Some applications of strong Markov property

We use (3.44) to derive some interesting properties of the BM.

(3.53) **Theorem:** (Reflection principle for BM)

Let $M_t = \sup_{s \le t} X_s$, $a > 0$, $b \le a$. Then

(a) $W_0(X_t \le b, M_t \ge a) = W_0(X_t \ge 2a - b)$, $t \ge 0$

(b) $W_0(M_t \ge a) = 2W_0(X_t \ge a) = W_0(|X_t| \ge a)$.

Proof: Let $H_a = \inf\{t \ge 0 : X_t = a\}$ be a $\mathcal{F}_t^+$-stopping time

Then $W_0(X_t \le b, M_t \ge a) = W_0(H_a \le t, X_t \le b)$

$= W_0\left(\{w \in C : H_a(w) \le t, X_{(t-H_a(w))_+}(\theta_{H_a}(w) \le b)\}\right)$

(3.54) **Lemma:** Let T be a $\mathcal{F}_t^+$-st. time, $h(w_1, w_2) \in b\mathcal{F} \otimes \mathcal{F}_T^+$, $x \in \mathbb{R}^d$.

$E_x[h(\theta_{T(w)}(w), w) \mid \mathcal{F}_T^+] = \int h(w_1, w) W_{X_{T(w)}}(dw_1)$, W_x-a.s on $\{T < \infty\}$

Proof: For $h = 1_{A_1}(w_1) 1_{A_2}(w_2)$, $A_1 \in \mathcal{F}$, $A_2 \in \mathcal{F}_T^+$, the strong MP gives

$E_x[h(\theta_T(w), w) 1_B] = E_x[1_B \int h(w_1, w) W_{X_{T(w)}}(dw_1)]$ $\forall B \in \mathcal{F}_T^+ \cap \{T < \infty\}$

The extension to arbitrary h follows by usual approximation arguments. $\square$

-(a) Applying (3.54) with $h(w_1, w_2) = 1\{X_{(t - H_a(w_2))_+}(w_1) \le b\}$.

which is $\mathcal{F} \times \mathcal{F}_{H_a}^+$-measurable ($\rightarrow$ Exercise) we have

$W_0(H_a \le t, X_t \le b) = E_0[H_a \le t, \underbrace{\tilde{W}}_{\text{copy of } W}{}_{\cancel{X_{H_a}} = a}(\tilde{X}_{(t-H_a)_+} \le b)]$

$\overset{\text{Symmetry}}{=} E_0[H_a \le t, \tilde{W}_a(\tilde{X}_{(t-H_a)_+} \ge 2a - b)]$

$\overset{(3.54)\text{ backwards}}{=} E_0[H_a \le t, X_t \ge 2a - b] = E_0[X_t \ge 2a - b]$.

-(b) For $a > 0$, $W_0[M_t \ge a] = W_0[H_a \le t] = W_0[H_a \le t, X_t > a] +$

$\underbrace{W_0[H_a \le t, X_t \le a]}_{\text{by (a) } W_0[X_t \ge a]} = 2W_0[X_t \ge a]$.

$\square$

(3.55) <u>Exercise</u>: (a) law of $H_a \ll$ Lebesgue and

$$W_0(H_a \in dt) = \frac{|a|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{a^2}{2t}\right\} 1\!\!1_{(0,\infty)}(t)\, dt$$

(b) The joint density of pair (X_t, M_t) for $t > 0$ is

$$W_0(X_t \in da, M_t \in db) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(2a-b)^2}{2t}\right\} 1\!\!1_{a<b,\, b>0}\, da\, db.$$

We push (3.53(b)) a bit further.

(3.56) <u>Corollary</u>: Let $Y_t = M_t - X_t$, then for every $t \geq 0$, under W_0, Y_t, $|X_t|$ and M_t have the same distribution given by (3.53(b))

<u>Proof</u>: By (3.56(b)), $M_t \overset{law}{=} |X_t|$. For Y_t, obviously $Y_t \geq 0$ W_0-a.s. Moreover, the time reversed process $B_s = X_{t-s} - X_t$, $s \in [0,t]$, is a Brownian motion ($\rightarrow$ Exercise [Use G. process characterisation of BM]), and $Y_t = \sup_{s \leq t} X_s - X_t = \sup_{0 \leq s \leq t} B_s$. Hence $Y_t \overset{law}{=} M_t$. □

Even more is true

(3.57) <u>Theorem</u>: Under W_0, both $(Y_t)_{t\geq 0}$, $(|X_t|)_{t \geq 0}$ are $\mathcal{F}_t^+$-Markov processes with transition semigroup $K_s(x, dy) = (p_s(x,y) + p_s(x,-y))\, dy$, where $p_s(x,y) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-y)^2}{2s}}$.
In particular, Y and $|X|$ have the same law under W_0.

(3.58) <u>Remark</u>: Even if $M_t \overset{law}{=} |X_t| \overset{law}{=} Y_t$, M as a process has not the same law as $|X|$ or Y. Indeed, M is increasing while $|X|$ and Y not. In fact, M_t is even not a Markov process.

(3.59) <u>Remark</u>: (3.57) claims that for $Z \in \{Y, |X|\}$,

$$W_0[Z_{t+s} \in dy \mid \mathcal{F}_t^+] = K_s(X_t, dy).$$

Proof: Consider the process $(|X_t|)_{t\geq 0}$ first. By the simple Markov prop.

$$W_0(|X_{t+s}|\in dy \mid \mathcal{F}_t^+) = W_0(|X_{t+s}|\in dy \mid X_t) =$$
$$W_0(X_{t+s}\in dy \mid X_t) + W_0(-X_{t+s}\in dy \mid X_t)$$
$$= p_s(X_t,y)\,dy + p_s(X_t,-y)\,dy = K_s(X_t,y)\,dy \qquad W_0\text{-a.s.}$$

Comparing this with Remark (3.59) implies that $(|X_t|_{t\geq 0})$ is $\mathcal{F}_t^+$-Markov process with trans. semigroup $K_s(x,dy)$.

For $(Y_t)_{t\geq 0}$ we write, with $s>0, t\geq 0, b\geq a, b\geq 0$.

$$W_0[X_{t+s}\leq a, M_{t+s}\leq b \mid \mathcal{F}_t^+] = W_0[X_{t+s}\leq a, M_t\leq b, \max_{0\leq u\leq s} X_{t+u}\leq b \mid \mathcal{F}_t^+]$$
$$= \mathbb{1}_{\{M_t\leq b\}} W_0[X_{t+s}\leq a, \max_{0\leq u\leq s} X_{t+s}\leq b \mid X_t],$$

by the simple Markov property again. The last expression is measurable w.r.t. the σ-field generated by (M_t, X_t). This implies that the process $\{(X_t, M_t)_{t\geq 0}\}$ is $\mathcal{F}_t^+$-Markov. Y_t is a function of (X_t, M_t) and thus

(3.60) $$W_0[Y_{t+s}\in A \mid \mathcal{F}_t^+] = W_0[Y_{t+s}\in A \mid X_t, M_t], \qquad A\in\mathcal{B}(\mathbb{R}).$$

Let us determine the transition probabilities of $(X_t, M_t)_{t\geq 0}$.

For $b>m\geq x$, $b\geq a$, $m\geq 0$ we have

$$W_0[X_{t+s}\in da, M_{t+s}\in db \mid X_t=x, M_t=m]$$

(3.61) $$= W_x[X_s\in da, M_s\in db] = W_0[X_s\in da-x, M_s\in db-x]$$
$$\overset{(3.55(a))}{=} \frac{2(2b-a-x)}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(2b-a-x)^2}{2s}\right\} da\,db.$$

and for $m\geq x$, $a\in\mathbb{R}$, $m\geq 0$ we have

$$W_0[X_{t+s}\in da, M_{t+s}=m \mid X_t=x, M_t=m] =$$

(3.62) $$= W_x[X_s\in da, M_s\leq m] = W_0[X_s\in da-x, M_s\leq m-x]$$
$$= \frac{1}{\sqrt{2\pi s}}\left[\exp\left\{-\frac{(a-x)^2}{2s}\right\} - \exp\left\{-\frac{(2m-a-x)^2}{2s}\right\}\right],$$

where the last equality follows by integrating (3.55(b)).

From (3.60)-(3.62) we get

$$W_0[Y_{t+s}\in dy \mid X_t=x, M_t=m] = \int_{(m,\infty)} W_0[X_{t+s}\in b-dy, M_{t+s}\in db \mid X_t=x, M_t=m]\,db$$
$$+ W_0[X_{t+s}\in m-dy, M_{t+s}=m \mid X_t=x, M_t=m] = \text{"algebra"} =$$
$$= K_s(m-x,dy) = K_s(Y_t,dy).$$

Hence $(Y_t)_{t\geq 0}$ is $\mathcal{F}_t^+$-Markov with the same trans. semigroup as $(|X_t|_{t\geq 0})$. The last claim follows from the fact that the semigroup determines the law of a Markov process $\square$

3.11 Properties of Brownian motion sample path

Recall that we already know that BM trajectories are α-Hölder for every $\alpha \in (0,\frac{1}{2})$ (3.16), but otherwise rather "rough": they are of infinite variation (3.24) and not differentiable at 0 (Exercise at p. 5/3 (+)). We now give another "roughness results".

(3.63) Theorem (Law of iterated logarithm) $(d=1)$ W_0-a.s.

(i) $$\limsup_{t\downarrow 0} \frac{X_t}{\sqrt{2t\log\log\frac{1}{t}}} = -\liminf_{t\downarrow 0} \frac{X_t}{\sqrt{2t\log\log\frac{1}{t}}} = 1,$$

(ii) $$\limsup_{t\nearrow\infty} \frac{X_t}{\sqrt{2t\log\log t}} = -\liminf_{t\nearrow\infty} \frac{X_t}{\sqrt{2t\log\log t}} = 1.$$

Proof: – (ii) follows from (i) by the time inversion invariance of BM (3.6 (c)). Using the reflection invariance (3.6 (a)), it is then sufficient to show $\limsup_{t\downarrow 0} \frac{X_t}{\sqrt{2t\log\log\frac{1}{t}}} = 1$.

Upper bound: Set $\varphi(t) = \sqrt{2t\log\log\frac{1}{t}}$. We approximate φ by piecewise constant functions and estimate X against those. To this end fix $\delta > 0$, $q\in(0,1)$ such that

(3.64) $$(1+\delta)^2 q > 1 \qquad (\text{i.e. } q\nearrow 1 \text{ as } \delta\downarrow 0)$$

For $n\geq 1$, set $t_n = q^n$ (i.e. $t_n\downarrow 0$) and

$A_n = \{ w\in C: \max\{X_t : t\in[t_{n+1},t_n]\} > (1+\delta)\varphi(t_{n+1})\}$.

Since $\varphi(t)$ is non-decreasing on some small interval $[0,T]$, the upper bound follows if we show that for every δ, A_n occurs W_0-a.s. only finitely many times.

To this end we write for n large

$$W_0(A_n) \leq W_0\Big(\sup_{s\leq t_n} X_s > (1+\delta)\varphi(t_{n+1})\Big) \overset{(\text{reflection p.})}{=} 2W_0\big(X_{t_n} \geq (1+\delta)\varphi(t_{n+1})\big)$$
$$\leq \sqrt{\frac{2}{\pi}}\,\frac{1}{x_n}\exp\Big\{-\frac{x_n^2}{2}\Big\}, \quad \text{with} \quad x_n = \frac{(1+\delta)\varphi(t_{n+1})}{\sqrt{t_n}}.$$

Note that

$$X_n = (1+\delta)\{2q \log((n+1)\log\tfrac{1}{q})\}^{1/2} = \{2\log\{(\alpha(n+1))^{\lambda}\}\}^{1/2}$$

with $\alpha = \log\frac{1}{q}$, $\lambda = q(1+\delta)^2 > 1$ by (3.64). Hence, for n large

$W_0(A_n) \le \sqrt{\frac{2}{\pi}} \frac{1}{(n+1)^{\lambda}\alpha^{\lambda}}$, which is summable. Borel-Cantelli lemma then yields the upper bound.

<u>Lower bound:</u> We fix $q \in (0,1)$, $\varepsilon \in (0,\frac{1}{2})$ small and set $t_n = q^n$. We will use the Gaussian estimate

(3.65) $$P[\mathcal{N}(0,1) \ge x] > \frac{x}{1+x^2}\frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x > 0.$$

Setting $x_n = (1-\varepsilon)\varphi(t_n)(t_n - t_{n+1})^{-1/2}$, we find for n large

(3.66) $$W_0(X_{t_n} - X_{t_{n+1}} > (1-\varepsilon)\varphi(t_n)) \overset{\text{scaling}}{=} W_0(X_1 > x_n) \overset{(3.65)}{\ge}$$
$$\ge (2\pi)^{-1/2} x_n (1+x_n^2)^{-1} \exp\{-\tfrac{x_n^2}{2}\}$$

Moreover $x_n = \frac{(1-\varepsilon)}{\sqrt{1-q}}\sqrt{2\log(n\log\frac{1}{q})} = \sqrt{\beta\log(\alpha n)}$ with $\alpha = \log\frac{1}{q}$, $\beta = 2(1-\varepsilon)^2/(1-q)$. Hence, for n large $\frac{x_n}{1+x_n^2} \ge \frac{1}{2x_n}$.

Assuming $q < \frac{\varepsilon^2}{4}$, i.e. $\beta < 2$, we get

$W_0(X_{t_n} - X_{t_{n+1}} > (1-\varepsilon)\varphi(t_n)) \ge \frac{c}{(\log n)^{1/2}} n^{-\beta/2}$. This is not summable and the events on the LHS of (3.66) are independent as n varies. Therefore, by the 2^{nd} Borel-Cantelli lemma, W_0-a.s., for infinitely many n, $X_{t_n} - X_{t_{n+1}} \ge (1-\varepsilon)\varphi(t_n)$.

From the upper bound applied to $(-X)$, we see that W_0-a.s. for n large $X_{t_n} \ge -(1+\varepsilon)\varphi(t_n)$, and thus W_0-a.s. for infinitely many n

$$X_{t_n} = X_{t_n} - X_{t_{n+1}} + X_{t_{n+1}} \ge (1-\varepsilon)\varphi(t_n) - (1+\varepsilon)\varphi(t_{n+1})$$
$$= \varphi(t_n)\left[1-\varepsilon-(1+\varepsilon)\frac{\varphi(t_n)}{\varphi(t_{n+1})}\right]$$

Using $q < \varepsilon^2/4$, we get $\lim_n \varphi(t_{n+1})/\varphi(t_n) = \sqrt{q} < \frac{\varepsilon}{2}$, so

$$\ge \varphi(t_n)[1-2\varepsilon].$$

Letting $\varepsilon \downarrow 0$ implies the lower bound. $\square$

(3.67) <u>Remarks:</u> Few related results

(1) <u>Lévy's modulus of continuity of BM.</u> (Lévy 1937)

W_0-a.s., $\limsup_{u \downarrow 0} \left(2u \log \frac{1}{u}\right)^{-1/2} \sup_{\substack{0 \le s < t \le 1 \\ |t-s| \le u}} |X_t - X_s| = 1.$

Proof of this claim is similar to the proof of the LIL, see e.g. Karatzas-Shreve, p. 114. Observe that $(2t \log\log \frac{1}{t})^{1/2}$ is replaced by "larger" $(2t \log \frac{1}{t})^{1/2}$. This is consequence of taking sup over the starting points $s \in [0,1]$. For fixed s, LIL holds by Markov prop.

(2) <u>LIL for small values of $\sup_{s \le t} |X_s|$</u> (Chung 1948)

W_0-a.s., $\liminf_{t \to \infty} \left(\frac{\log\log t}{t}\right)^{1/2} \sup_{s \le t} |X_s| = \frac{\pi}{2\sqrt{2}} < 1.$

Observe also, by LIL, $\limsup \left(\frac{1}{2t \log\log t}\right)^{1/2} \sup_{s \le t} |X_s| = 1.$

(3) <u>Strassen theorem</u> (1964) gives a functional version of LIL. The set of limit points of $\left(\frac{X_{ut}}{\sqrt{2t \log\log t}}\right)_{0 \le u \le 1}$ as $t \to \infty$

coincides with $\{ f \in C([0,1], \mathbb{R}) : f(u) = \int_0^u g(s)\,ds,\ g \in L^2([0,1]),\ \|g\|_{L^2} \le 1 \}.$

(3.68) <u>Theorem:</u> ("Converse" to (3.16)). Let $\alpha \ge \frac{1}{2}$. Then, for $0 \le a < b < \infty$

W_0-a.s., $\sup_{a \le s < t \le b} \frac{|X_s - X_t|}{|s-t|^\alpha} = +\infty.$

<u>Proof:</u> By the LIL and Markov property

W_0-a.s., $\forall s \in \mathbb{Q} \cap [0,\infty)$, $\limsup_{u \downarrow 0} \frac{|X_{s+u} - X_s|}{\sqrt{2u \log\log \frac{1}{u}}} = 1.$ Hence,

W_0-a.s., $\forall s \in \mathbb{Q} \cap [0,\infty)$, $\limsup_{u \downarrow 0} \frac{|X_{s+u} - X_s|}{u^\alpha} = \infty$

and the claim follows $\square$

Chapter IV. – CONTINUOUS-TIME MARTINGALES

We develop here the theory of martingales in continuous-time, and of the theory of st. processes in general.

(4.1) Definition: We say that a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ satisfies the usual conditions if

(i) $\mathcal{G}_0$ contains all P-null sets, i.e. $\Gamma \subset N \in \mathcal{G}$, $P(N) = 0 \Rightarrow \Gamma \in \mathcal{G}_0$

(ii) $(\mathcal{G}_t)_{t\geq 0}$ – is right continuous, i.e. $\mathcal{G}_t = \bigcap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon}$, $\forall t \geq 0$.

(4.2) Example: Consider the canonical BM $(C, \mathcal{F}, W_0)$. For $t \geq 0$, set
$$\overline{\mathcal{F}}_t = \{A \subset C : \exists B \in \mathcal{F}_t \text{ s.t. } A \Delta B \text{ is } W_0\text{-null}\}.$$

(4.3) Lemma (i) $\mathcal{F}_t^+ \subset \overline{\mathcal{F}}_t$

(ii) $(\overline{\mathcal{F}}_t)$ is right-continuous

(iii) $(C, \mathcal{F}, (\overline{\mathcal{F}}_t)_{t\geq 0}, W_0)$ satisfies the usual conditions.

Proof: – (i) Observe, that for every $A \in \mathcal{F}$ there is $Y \in b\mathcal{F}_t$ such that $E_0[1\!\!1_A \mid \mathcal{F}_t^+] = Y$, W_0-a.s. Indeed, for $A = \{X_{t_0} \in D_0, \dots, X_{t_k} \in D_k, X_{s_1} \in D_{k+1}, \dots, X_{s_m} \in D_{k+m}\}$ with $0 = t_0 < \dots < t_k = t < s_1 < \dots < s_m$, $D_i \in \mathcal{B}(\mathbb{R}^d)$, by the simple M. prop.
$$E_0[1\!\!1_A \mid \mathcal{F}_t^+] \overset{W_0\text{-a.s.}}{=} 1\!\!1\{X_{t_0} \in D_0, \dots, X_{t_k} \in D_k\} W_{X_t}[X_{s_1 - t} \in D_{k+1}, \dots, X_{s_m - t} \in D_{k+m}]$$
which is $\mathcal{F}_t$-measurable. Dynkin's type arguments then yield the claim for general $A \in \mathcal{F}$. (This can actually be seen as a generalisation of the 0-1 law). Hence, for $A \in \mathcal{F}_t^+$, $1\!\!1_A = E[1\!\!1_A \mid \mathcal{F}_t^+] = Y$, W_0-a.s. for some $Y \in b\mathcal{F}_t$. Hence, $1\!\!1_A = 1\!\!1_{\{Y=1\}}$, W_0-a.s. with $\{Y = 1\} \in \mathcal{F}_t$ and (i) follows.

– (ii) Let $\overline{\mathcal{F}}_t^+ = \bigcap_{\varepsilon>0} \overline{\mathcal{F}}_{t+\varepsilon}$ and let $A \in \overline{\mathcal{F}}_t^+$. We need to show that $A \in \overline{\mathcal{F}}_t$. By assumption, $A \in \overline{\mathcal{F}}_{t+\frac{1}{n}}$ for each $n \geq 1$. Hence, there is $B_n \in \mathcal{F}_{t+\frac{1}{n}}$ s.t. $W_0(A \Delta B_n) = 0$. Define $B \in \mathcal{F}_t^+$ via $1\!\!1_B = \limsup_n 1\!\!1_{B_n}$ (prove that $B \in \mathcal{F}_t^+$!) Then $W_0(A \Delta B) = 0$. By (i), there is $C \in \mathcal{F}_t$

such that $W_0(C \Delta B) = 0$, and thus $W_0(A \Delta C) = 0$, i.e. $A \in \bar{\mathcal{F}}_t$ as required

– (iii) (4.1(i)) follows from the definition of $\bar{\mathcal{F}}_t$, (4.1(ii)) follows from (ii) □

From now on, we assume that $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ satisfies the usual conditions, if not said otherwise.

If $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ satisfies the usual conditions, the following theorem allows to construct numerous stopping times, cf. also (3.33) – (3.35).

(4.4) <u>Theorem</u> Let $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ satisfy the usual conditions, and let $A \subset \mathbb{R}_+ \times \Omega$ be a progressively measurable set (i.e. $X_t(\omega) = 1\!\!1_A(t,\omega)$ is prog. measurable process). Let further D_A be the debut of A

$$D_A(\omega) = \inf\{t \geq 0 : (t,\omega) \in A\}.$$

Then D_A is a st. time.

<u>Proof</u>: Uses the following difficult theorem of measure theory

(4.5) <u>Theorem</u>: Let $(\Lambda, \mathcal{H}, \Pi)$ be a complete probability space and let $H \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{H}$. Then its projection $p(H) = \{\omega \in \Lambda : \exists t \geq 0, (t,\omega) \in H\}$ is in $\mathcal{H}$

To prove (4.4), we now fix $t \geq 0$, take $(\Lambda, \mathcal{H}, \Pi) = (\Omega, \mathcal{G}_t, P)$ and $H = ([0,t) \times \Omega) \cap A$. Then $H \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{H}$, since A is prog. measurable, and thus

$$p(H) = \{\omega \in \Omega : \exists s < t, (s,\omega) \in A\} = \{D_A < t\} \in \mathcal{G}_t.$$

As $\mathcal{G}_t$ is right continuous, D_A is st. time by (3.37). □

4.2 Martingales

(4.6) Definition: A $\mathbb{R}$-valued process $(X_t)_{t\geq 0}$ is called $\mathcal{G}_t$-martingale if it is $\mathcal{G}_t$-adapted, $X_t \in L^1(P)$ $\forall t \geq 0$, and for every $0 \leq s \leq t < \infty$

(4.7) $$E(X_t \mid \mathcal{G}_s) = X_s \quad P\text{-a.s.}$$

It is called sub-/super-martingale if (4.7) holds with "$\geq$"/"$\leq$".

(4.8) Exercise Let $(Z_t)_{t\geq 0}$ be a process with independent increments wrt $\mathcal{G}_t$ that is for $0 \leq s \leq t$, $Z_t - Z_s$ is independent of $\mathcal{G}_s$, Z is adapted. (e.g. BM has independent increments wrt $\mathcal{F}_t$ or $\mathcal{F}_t^+$ or $\overline{\mathcal{F}}_t$). Then

(i) If $Z_t \in L^1$ $\forall t$, then $\hat{Z}_t = Z_t - E[Z_t]$ is a mtg.

(ii) If $Z_t \in L^2$ $\forall t$, then $Z_t' = Z_t^2 - E[Z_t^2]$ – " –

(iii) If for $\theta \in \mathbb{R}$ $E[e^{\theta Z_t}] < \infty$ $\forall t$, then $\bar{Z}_t = \frac{e^{\theta Z_t}}{E[e^{\theta Z_t}]}$ is a mtg.

(4.9) Examples: Using (4.8) it follows that if B_t is a BM, $\mathcal{G}_t = \sigma(B_s, s \leq t)$, then B_t, $B_t^2 - t$, $\exp\{\theta B_t - \frac{1}{2}\theta^2 t\}$ are mtgs. Also, for $f \in L^2_{loc}(\mathbb{R}_+, dt)$,
$$Z_t = \int_0^t f(s)\,dB_s \quad \left(= G(f\,1\!\!1_{[0,t]})\right)$$
has independent increments, as follows from G. measures properties. I.e., $\int_0^t f(s)\,dB_s$, $\left(\int_0^t f(s)\,dB_s\right)^2 - \int_0^t f(s)^2\,ds$, $\exp\left(\theta \int_0^t f(s)\,dB_s - \frac{\theta^2}{2}\int_0^t f(s)^2\,ds\right)$ are martingales.

(4.10) Exercise Let N_t be the standard Poisson process, $\mathcal{G}_t = \sigma(N_s, s \leq t)$. Then N_t has indep. increments and $N_t - t$, $(N_t - t)^2 - t$ are mtgs.

(4.11) Exercise: (a) Let X_t be $\mathcal{G}_t$-(sub-)mtg, and $f: \mathbb{R} \to \mathbb{R}$ (increasing) convex. If $f(X_t) \in L^1$ $\forall t$, then $f(X_t)$ is submartingale.

(b) Let X_t be $\mathcal{G}_t$-sub/super-mtg. Then $\sup_{s \leq t} E[|X_s|] < \infty$ for all $t \geq 0$.

We now generalise some results on discrete time martingales to continuous time

(4.12) Theorem (Doob's inequality). Let X be a right-continuous submartg. Then
$$\lambda P[\sup_{0\le s\le t} X_s \ge \lambda] \le E[X_t^+] := E[X_t \vee 0], \ \lambda > 0, t \ge 0.$$
Proof: Recall that a discrete-time submartg $(Y_m)_{m\ge 0}$ satisfies the discrete version of Doob's inequality, $\lambda P[\sup_{m\le n} Y_m \ge \lambda] \le E[Y_n^+]$. for $\lambda > 0$, $n \in \mathbb{N}$. Observe also that for every $t_0 \le t_1 \le \dots \le t_k = t$ the process $Y_k = X_{t_k}$ is a discrete-time sub-martg w.r.t. filtration $\mathcal{H}_k = \mathcal{G}_{t_k}$. Hence, for every countable set $D \subset \mathbb{R}_+$, which we can write as increasing limit of such finite sequences,
$$\lambda P[\sup_{s\in[0,t]\cap D} X_s \ge \lambda] \le E[X_t^+].$$
If one assumes that X is right-continuous and takes D to be dense in $\mathbb{R}_+$, one obtains the claim of the theorem $\square$

In an analogous way, one proves:

(4.13) Theorem [Doob's L^p-inequality]. Let X be a right-continuous non-negative submartg. Then
$$E[(\sup_{s\le t} X_s)^p]^{1/p} \le \frac{p}{p-1} E[X_t^p]^{1/p}, \ 0 \le t < \infty, p \in (1,\infty).$$

(4.14) Exercise: Prove the theorem

Upcrossings. For $f: T \to \mathbb{R}$, $T \subset \mathbb{R}_+$, $a < b$, let $U_{ab}^f(T)$ be the number of upcrossings of $[a,b]$ by f in T, i.e. the maximal k such that there is $s_1 < t_1 < \dots < s_k < t_k$ with $s_i, t_i \in T$, $f(s_i) < a$, $f(t_i) > b$.
For a discrete time submartg Y_n we know

(4.15) $$(b-a)\, E U_{ab}^Y(\{0,\dots,n\}) \le E((Y_n - a)^+).$$

As consequence,

(4.16) Proposition: Let $(X_t)_{t\geq 0}$ be a submartingale, D a countable subset of $\mathbb{R}_+$, then
$$(b-a)\, E\left[M^X_{ab}([0,t]\cap D)\right] \leq E\left[(X_t-a)_+\right].$$

Recall also discrete-time martingale convergence theorems

(4.17) Theorem (a) If Y_n is submartingale and $\sup_n E Y_n^+ < \infty$, then $Y_n \xrightarrow{n\to\infty} Y$ a.s.

(b) If $(Y_n)_{n\in -\mathbb{N}}$ is a submartingale indexed by $-\mathbb{N}$, $E Y_0^+ < \infty$, then $Y_n \xrightarrow{n\to -\infty} Y$ a.s. Moreover, if $\sup_n E|Y_n| < \infty$ then $Y_n \xrightarrow{n\to\infty} Y$ in L^1

4.3 Martingales regularity

We now show that martingale trajectories posses certain regularity.

(4.18) Theorem: Let $(X_t)_{t\geq 0}$ be a submartingale and $D\subset\mathbb{R}_+$ dense, countable.

(i) For P-a.e. ω, the map $D\ni s\mapsto X_s(\omega)$ admits for every $t\in[0,\infty)$ a finite limit from the right $X_{t+}(\omega)$, and for $t\in(0,\infty)$ a finite limit from the left $X_{t-}(\omega)$.

(ii) For every $t\geq 0$, $X_{t+}\in L^1$ and
$$X_t \leq E[X_{t+}\mid \mathcal{F}_t]$$
with equality if $t\mapsto E[X_t]$ is right-cont. in t (in particular when X is martingale). Process $(X_{t+})_{t\geq 0}$ is then submartingale wrt. $\mathcal{G}_t^+$ (and a martingale, if X is a martingale).

Proof. (i) By Doob's inequality, for $T>0$, $\lim_{\lambda\to\infty} P\left[\sup_{D\cap[0,T]} |X_s| \geq \lambda\right]=0$, that is $\sup_{D\cap[0,T]} |X_s| < \infty$, a.s.. The upcrossing inequality yields then that a.s. for every $a<b\in\mathbb{Q}$, $U^X_{ab}(D\cap[0,T]) < \infty$. From this (i) follows easily.

– (ii) as X_{t+} is defined only a.s, we set $X_{t+}(\omega)=0$ if the right-limit along D does not exists. Let fix t and a sequence $t_k\downarrow t$, $t_k\in D$. Then, $X_{t+} = \lim_{n\to\infty} X_{t_k}$, a.s. For $k\leq 0$, set $Y_k = X_{t_{-k}}$.

Then $(Y_k)_{k\in -\mathbb{N}}$ is a submartingale indexed by $-\mathbb{N}$ w.r.t. $\mathcal{H}_k = \mathcal{G}_{t_{-k}}$, and also $\sup_k E|Y_k| < \infty$, so by (4.17(a)), $X_{t_k} \to X_{t+}$ in L^1. Thanks to this convergence, we may take a limit in an inequality $X_t \le E[X_{t_n} | \mathcal{G}_t]$ and find $X_t \le E[X_{t+} | \mathcal{G}_t]$.
Also, due to L^1-convergence, $E[X_{t+}] = \lim E[X_{t_n}] = E[X_t]$, where the last equality holds only if $t \mapsto E[X_t]$ is right-cont. In this case thus $X_t = E[X_{t+} | \mathcal{G}_t]$. Finally, X_{t+} is in $\mathcal{G}_t^+$, and for $s<t$, $s_n \downarrow s$ in D, $s_n < t_n$ and $A \in \mathcal{G}_s^+$

$$E[X_{s+} 1\!\!1_A] = \lim E[X_{s_n} 1\!\!1_A] \le \lim E[X_{t_n} 1\!\!1_A] = E[X_{t+} 1\!\!1_A] =$$
$$= E[E[X_{t+} | \mathcal{G}_{s+}] 1\!\!1_A]$$

Hence, $X_{s+} \le E[X_{t+} | \mathcal{G}_{s+}]$, so X_{t+} is submartingale $\square$

(4.19) <u>Theorem</u>: Assume that $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ satisfies the usual conditions. If $(X_t)_{t\ge 0}$ is a submartingale and $t \mapsto E[X_t]$ is right-continuous, then X has a modification which is also $\mathcal{G}_t$-submartingale whose trajectories are càdlàg.

<u>Proof</u>: Let D be as in (4.18). Let N be a null-set, where (i) of (4.18) does not hold and set

$$Y_t(\omega) = \begin{cases} X_{t+}(\omega) & \text{if } \omega \notin N \\ 0 & \text{if } \omega \in N \end{cases}$$

Y has right-continuous trajectories. Indeed, for $\omega \notin N$, $t \ge 0$, $\varepsilon > 0$,

$$\sup_{[t,t+\varepsilon)} Y_s(\omega) \le \sup_{(t,t+\varepsilon]\cap D} X_s(\omega), \quad \inf_{[t,t+\varepsilon)} Y_s(\omega) \ge \inf_{(t,t+\varepsilon]\cap D} X_s(\omega) \text{ and}$$
$$\lim_{\varepsilon\downarrow 0} \Big(\sup_{(t,t+\varepsilon]\cap D} X_s(\omega)\Big) = \lim_{\varepsilon\downarrow 0} \Big(\inf_{(t,t+\varepsilon]\cap D} X_s(\omega)\Big) = X_{t+}(\omega) = Y_t(\omega).$$

Similarly, we show that Y has left limits. Moreover, as $\mathcal{G}_t = \mathcal{G}_t^+$,
$X_t = E[X_{t+} | \mathcal{G}_t] = E[X_{t+} | \mathcal{G}_t^+] = X_{t+} = Y_t$ a.s., so Y is a modif. of X. As $\mathcal{G}_t$ is complete, $Y_t \in \mathcal{G}_t$ and it is submartingale $\square$.

4.4. Stopping theorems:

This is analogous to (4.17 (a))

(4.20) Exercise: If X is a right-continuous submartingale, $\sup_t E|X_t| < \infty$, then there is r.v. $X_{\infty-} \in L^1$ s.t. $\lim_{t \uparrow \infty} X_t = X_{\infty-}$ a.s.

(4.21) Proposition: Let $(X_t)_{t \geq 0}$ be a right-continuous martingale. TFAE

(i) There is $X_\infty \in L^1$ s.t. $X_t = E[X_\infty \mid \mathcal{G}_t]$, $t \geq 0$.

(ii) X_t converges as $t \to \infty$ a.s. and in L^1 to a r.v. $X_{\infty-}$

(iii) $(X_t : t \geq 0)$ is uniformly integrable.

Proof: (i) $\Rightarrow$ (iii) exercise

(iii) $\Rightarrow$ (ii) By (4.20) $X_t \to X_{\infty-}$ a.s. and by UI also in L^1.

(ii) $\Rightarrow$ (i) We have $X_s = E[X_t \mid \mathcal{G}_s]$. Take $t \uparrow \infty$. Then by (ii) also $X_s = E[X_{\infty-} \mid \mathcal{G}_s]$, i.e. (i) holds with $X_\infty = X_{\infty-}$ $\square$

(4.22) Theorem: Let X be a right-continuous martingale s.t. $X_t = E[X_\infty \mid \mathcal{G}_t]$. Let S, T be two st. times with $S \leq T$. Then $X_S, X_T \in L^1$ and

$$X_S = E[X_T \mid \mathcal{G}_S],$$

where we use convention $X_T = X_\infty$ on $\{T = \infty\}$.

In particular, $X_S = E[X_\infty \mid \mathcal{G}_S]$.

Proof Idea: Follows from the discrete-time optional stopping theorem by a discretisation argument, i.e. by setting

$$T_n = \inf\{t \in \tfrac{\mathbb{N}}{2^n} : t > T\}, \text{ i.e. } T_n \downarrow T$$

and using the right-continuity + backward martingale theorem. $\square$

(4.23) Corollary: If X is a right-continuous UI martingale and T a st. time, then $Y_t = X_{t \wedge T} =: X_t^T$ is also UI martingale, and $X_t^T = E[X_T \mid \mathcal{G}_t]$.

(4.24) Exercise: Prove this.

4.5. "Irregularity" of continuous martingales

We have seen that mtg. trajectories can be modified to be cadlag. We will now show that in continuous case they are quite irregular.

Recall that variation of $w \in C$ in $[s,t]$ is given by

(4.25) $$V_{s,t}(w) = \sup_{\substack{s=t_0<\dots<t_n=t \\ \text{rationals}}} \sum_{i=1}^{n} |w(t_i) - w(t_{i-1})|$$

(4.26) Lemma: Let X be a cont. mtg., $X_0 = 0$ and P-a.s. $X(\omega)$ has a finite variation on every $[0,t]$, $t \geq 0$. Then $X \equiv 0$ P-a.s.

Proof: Let S_n be stopping times

$$S_n = \inf\{s \geq 0 : |X_s| \geq n\} \wedge \inf\{s \geq 0 : V_{0,s}(X) \geq n\} \wedge n$$

By continuity of X and the assumption of finite variation, $S_n \nearrow \infty$. Moreover, $X_{t \wedge S_n}$ is a bdd. mtg for every $n \geq 0$ ($\to$ exercise). It is thus sufficient to show (4.25) for bdd. cont. mtgs. with bdd variation.

For those, by mtg. property

$$E[X_t^2] = E\Big[\Big\{\sum_{k=1}^{2^\ell} (X_{t\frac{k+1}{2^\ell}} - X_{t\frac{k}{2^\ell}})\Big\}^2\Big] = E\Big[\sum_{k=1}^{2^\ell} (X_{t\frac{k+1}{2^\ell}} - X_{t\frac{k}{2^\ell}})^2\Big]$$

$$\leq E\Big[\underbrace{\sup_{1\leq k\leq 2^\ell} |X_{t\frac{k+1}{2^\ell}} - X_{t\frac{k}{2^\ell}}|}_{\xrightarrow{\ell\to\infty} 0 \text{ by continuity}} \underbrace{\sum_{k=1}^{2^\ell} |X_{t\frac{k+1}{2^\ell}} - X_{t\frac{k}{2^\ell}}|}_{\leq V_{0,t}(X) < \infty}\Big] \xrightarrow{\ell\to\infty} 0$$

It follows that P-a.s. for all $t \in \mathbb{Q}$, $X_t = 0$ so $X \equiv 0$ P-a.s. by continuity $\square$

4.6 Functions and processes of finite variation

Let $a : [0,T] \to \mathbb{R}$ be of finite variation, i.e. $V_{0,T}(a) < \infty$. As follows from measure theory, there is a finite signed measure μ on $([0,T], \mathcal{B}([0,T]))$ associated with a by

(4.27) $$\mu([0,t]) = a(t) \qquad \forall t \le T.$$

μ is the Lebesgue - Stieltjes measure of a. μ can be written as $\mu = \mu_+ - \mu_-$ for two finite positive measures with disjoint supports. This decomposition is unique. It follows that a is a difference of two increasing functions, $\mu_+([0,t])$ and $\mu_-([0,t])$. If a is continuous and $a(0)=0$, then μ_+ and μ_- have no atoms, so also these functions are continuous. Finally, we set $|\mu| = \mu_+ + \mu_-$.

If $f : [0,T] \to \mathbb{R}$ is measurable and $\int_{[0,T]} |f| \, d|\mu| < \infty$, we set
$$\int_0^T f(s)\,da(s) = \int_{[0,T]} f(s)\,\mu(ds)$$
$$\int_0^T f(s)\,|da(s)| = \int_{[0,T]} f(s)\,|\mu|(ds).$$

(4.28) Observe that the map
$$t \longmapsto \int_0^t f(s)\,da(s)$$
is also of finite variation, the associated L-S measure is $\mu'(ds) = f(s)\,\mu(ds)$.

(4.29) <u>Exercise</u>: Let $f : [0,T] \to \mathbb{R}$ be a cont function, and $0 = t_0^n < \dots < t_{k_n}^n = T$ a sequence of subdivisions with stepsize tending to 0. Then
$$\int_0^T f(s)\,da(s) = \lim_{n\to\infty} \sum_{i=1}^{k_n} f(t_{i-1}^n)\left(a(t_i^n) - a(t_{i-1}^n)\right)$$

(4.30) <u>Definition</u>: A function $a : \mathbb{R}_+ \to \mathbb{R}$ is called (locally) of finite variation if it is of finite variation on every $[0,T]$, $T \ge 0$. In this case one easily defines $\int_0^\infty f(s)\,da(s)$ for all f such that $\int_0^\infty |f|\,|da(s)| = \sup_T \int_0^T |f|\,|da(s)| < \infty$.

Let $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ be a probability space satisfying the usual conditions.

(4.31) Definition A process of finite variation $(A_t)_{t\geq 0}$ is a adapted stoch. process whose trajectories are of finite variation as in (4.30). A is called increasing, if its trajectories are increasing.

(4.32) Remark: We will mostly consider continuous processes with finite variation with $A_0 = 0$. In this case the associated (random) L-S measure has no atoms.

(4.33) Proposition (integral w.r.t. A). Let A be a process of finite variation and H a progressively measurable process such that

$$\forall t\geq 0 \forall \omega\in\Omega, \quad \int_0^t |H_s(\omega)|\,|dA_s(\omega)| < \infty.$$

Then the process $H\cdot A$ given by

$$(H\cdot A)_t^{(\omega)} = \int_0^t H_s^{(\omega)} dA_s(\omega)$$

is also of finite variation, and continuous if A is.

Proof. The fact that the trajectories of $H\cdot A$ are of finite variation (and continuous) follows directly from (4.28). We only need to show that $H\cdot A$ is adapted, that is $(H\cdot A)_t \in \mathcal{G}_t$. This is easy if $H_s(\omega) = 1\!\!1_{(u,v]}(s)\, 1\!\!1_\Gamma(\omega)$ with $(u,v]\subset[0,t]$, $\Gamma\in\mathcal{G}_t$. and thus, by Dynkin's argument, for $H_s(\omega) = 1\!\!1_G(s,\omega)$ with $G\in\mathcal{B}([0,t])\otimes\mathcal{G}_t$. General prog. measurable H can be then written as a limit of step processes $H^n = \sum_{i=1}^{k_n} c_i 1\!\!1_{G_i}$, $G_i\in\mathcal{B}([0,t])\otimes\mathcal{G}_t$, $c_i\in\mathbb{R}$, with $|H^n|\leq|H|$, which insures that $(H^n\cdot A)(\omega)\xrightarrow{n\to\infty}(H\cdot A)(\omega)$ for every ω, by DCT. Hence $(H\cdot A)$ is adapted. $\square$.

4.7 Local martingales

Let $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ satisfy the usual conditions. If T is stopping time and X a continuous stoch. process, we define stopped process X^T

(4.34) $$X^T_t = X_{t \wedge T}, \quad t \geq 0.$$

(4.35) <u>Definition</u>. A adapted continuous process $(M_t)_{t\geq 0}$ s.t. $M_0 = 0$ is called (continuous) local martingale if there exists an increasing sequence T_n of stopping times such that $T_n \uparrow \infty$ and for every n, M^{T_n} is a uniformly integrable martingale.

If $M_0 \neq 0$, we say that M is (cont.) local martingale, if $M = M_0 + N_t$ for a local martingale N_t with $N_0 = 0$.

In both cases, we say that (T_n) reduces M and M^{T_n} is UI martingale.

(4.36) <u>Remark</u>. It is not necessary that $M_t \in L^1$ for a local mrtg.

(4.37) <u>Properties</u>: (i) A continuous mrtg is a local mrtg (take $T_n = n$)

(ii) In definition (4.35) of the local mrtg (started at 0) we may replace "UI martingale" by "martingale" (as we can replace T_n by $T_n \wedge n$).

(iii) If M is a local mrtg and T a st. time, then M^T is local mrtg.

(iv) If (T_n) reduces M and S_n is a seq. of stop. times with $S_n \uparrow \infty$, then $(S_n \wedge T_n)$ reduces M.

(v) The space of local mrtgs is a vector space

(4.38) <u>Exercise</u>: Prove (v) by applying (iv)

(4.39) **Proposition** (i) A non-negative local mtg M with $M_0 \in L^1$ is a supermtg.
(ii) If M is local mtg and $|M_t| \le Z$ $\forall t$ with $Z \in L^1$, then M is mtg.
(iii) If M is local mtg, $M_0 = 0$, then $T_n = \inf\{t : |M_t| \ge n\}$ reduces M.

Proof. - (i) Let $M_t = M_0 + N_t$ and let (T_n) reduces N. Then for $s \le t$, $n \in \mathbb{N}$

$$N_{s \wedge T_n} = E[N_{t \wedge T_n} \mid \mathcal{G}_s]$$

and thus, adding M_0 which is in L^1,

$$M_{s \wedge T_n} = E[M_{t \wedge T_n} \mid \mathcal{G}_s]$$

As $M \ge 0$, by Fatou's lemma for conditional expectation

$$M_s \ge E[M_t \mid \mathcal{G}_s].$$

Taking $s = 0$, we see also $E[M_t] \le E[M_0]$, i.e. $M_t \in L^1$ and M is thus supermtg.

- (ii) Follows by the same steps, using DCT instead of Fatou.
- (iii) The fact that M^{T_n} is a UI mtg follows from (ii), $T_n \uparrow \infty$ from the continuity. $\square$

(4.40) *Lemma*: Let M be a cont. local mtg, $M_0 = 0$. If M is process of finite variation, then M is indistinguishable from 0.

Proof: This is a small extension of (4.26). $\square$

4.8. Quadratic variation of local martingales

Recall that in Theorem (3.23) we proved that quadratic variation of BM is t. Moreover, by (4.9), $B_t^2 - t$ is a mart. We now prove that for arbitrary continuous local martingale M there is an increasing process $\langle M \rangle$ which has similar properties as the process "$t \mapsto t$" has in the case of BM. This process will play a key role when developing the stochastic integral.

(4.41) Theorem Let $(M_t)_{t\geq 0}$ be a (continuous) local mart on $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ satisfying the usual conditions. Then there exists an essentially unique (i.e. defined modulo indistinguishability) increasing process adapted process $(\langle M \rangle_t)_{t\geq 0}$ with $\langle M \rangle_0 = 0$ such that $M_t^2 - \langle M \rangle_t$ is a local mart. Moreover, for every $T>0$, if $0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = T$ is a sequence of refining partitions with mesh tending to 0,

(4.42) $$\langle M \rangle_T = \lim_{n\to\infty} \sum_{i=1}^{k_n} \left(M_{t_i^n} - M_{t_{i-1}^n} \right)^2 \quad \text{in probability}$$

$\langle M \rangle$ is called quadratic variation of the local mart M.

Proof: – Uniqueness: Let A, B be two adapted continuous increasing processes with $A_0 = B_0 = 0$ and $M_t^2 - A_t$, $M_t^2 - B_t$ being local marts. Then $(M_t^2 - A_t) - (M_t^2 - B_t) = B_t - A_t$ is again a cont. local mart. Moreover, it is of finite variation, so by (4.40) $B_t - A_t \equiv 0$ P-a.s, yielding the essential uniqueness.

– Existence: We are going to use (4.42) to construct $\langle M \rangle$. To this end, we fix $T>0$ and $(t_i^n)_{i=1,\ldots,k_n}$, $n\in\mathbb{N}$, as in the theorem and define

$$I_t^n = \sum_{i=1}^{k_n} M_{t_{i-1}^n} \left(M_{t_i^n \wedge t} - M_{t_{i-1}^n \wedge t} \right), \quad n\in\mathbb{N},\ t\in[0,T].$$

Case 1: $M_0 = 0$, M is bounded martingale.

The role of I^n is the following: First, for every $t \in [0,T]$, $n \in \mathbb{N}$.

(4.43)
$$M_t^2 - 2I_t^n = \Big(\sum_{i=1}^{\ell} (M_{t_i^n} - M_{t_{i-1}^n}) + (M_t - M_{t_\ell^n})\Big)^2 - 2\sum_{i=1}^{\ell} M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n}) - 2M_{t_\ell^n}(M_t - M_{t_\ell^n})$$
$$\overset{(M_0=0)}{=} \sum_{i=1}^{\ell} (M_{t_i^n} - M_{t_{i-1}^n})^2 + (M_t - M_{t_\ell^n})^2,$$

where ℓ is such that $t_\ell^n \le t < t_{\ell+1}^n$, which links I^n with (4.42).

Moreover, for every n, I_t^n, $t \le T$, is a continuous martingale which makes it suitable for computations. To see that, let $s \le t \le T$, $A \in \mathcal{G}_s$,

(4.44) and verify $\mathbb{E}\big[M_{t_{i-1}^n}(M_{t_i^n \wedge t} - M_{t_{i-1}^n \wedge t}) 1\!\!1_A\big] = \mathbb{E}\big[M_{t_{i-1}^n}(M_{t_i^n \wedge s} - M_{t_{i-1}^n \wedge s}) 1\!\!1_A\big]$

separately for $t_{i-1}^n \le s$, $s < t_{i-1}^n \le t$, $t < t_{i-1}^n$.

- $t < t_{i-1}^n$: both sides of (4.44) are trivially 0.
- $s < t_{i-1}^n \le t$. LHS of (4.44) $= \mathbb{E}\Big[1\!\!1_A M_{t_{i-1}^n} \underbrace{\mathbb{E}\big[M_{t_i^n \wedge t} - M_{t_{i-1}^n \wedge t} \mid \mathcal{G}_{t_{i-1}^n}\big]}_{=0}\Big] = 0$
$$= \text{RHS of (4.44)}.$$
- $t_{i-1}^n \le s$: LHS of (4.44) $= \mathbb{E}\big[1\!\!1_A M_{t_{i-1}^n} \mathbb{E}[M_{t_i^n \wedge t} - M_{t_{i-1}^n \wedge t} \mid \mathcal{G}_s]\big]$
$\overset{\text{mart.}}{=} \mathbb{E}\big[1\!\!1_A M_{t_{i-1}^n}(M_{t_i^n \wedge s} - M_{t_{i-1}^n \wedge s})\big] = \text{RHS of (4.44)}$.

Using the mart. property of I^n we now show that

(4.45)
$$\lim_{m,n\to\infty} \mathbb{E}\big[(I_T^n - I_T^m)^2\big] = 0.$$

We assume $m < n$, with $\rho_k = t_k^m$, $\tau_\ell = t_\ell^n$. Then, by assumption $\{\rho_k\} \subset \{\tau_\ell\}$

(4.46)
$$I_T^n - I_T^m = \sum_{\ell=1}^{k_n} M_{\tau_{\ell-1}}(M_{\tau_\ell} - M_{\tau_{\ell-1}}) - \sum_{k=1}^{k_m} M_{\rho_{k-1}}(M_{\rho_k} - M_{\rho_{k-1}})$$
$$= \sum_{k,\ell} 1\!\!1\{\rho_{k-1} \le \tau_{\ell-1} < \rho_k\} M_{\tau_{\ell-1}}(M_{\tau_\ell} - M_{\tau_{\ell-1}})$$
$$- \sum_{k,\ell} 1\!\!1\{ \text{ -- " -- } \} M_{\rho_{k-1}}(M_{\tau_\ell} - M_{\tau_{\ell-1}})$$
$$= \sum_{k,\ell} 1\!\!1\{\rho_{k-1} \le \tau_{\ell-1} \le \rho_k\}(M_{\tau_{\ell-1}} - M_{\rho_{k-1}})(M_{\tau_\ell} - M_{\tau_{\ell-1}}) =: \sum_{k,\ell} a_{k,\ell}.$$

(4.47) We claim that $a_{k,l}$ are orthogonal in $L^2(P)$ (they are bdd)
Indeed, for $k<k'$ and $l\geq 0$, we have $a_{k,l}(\omega)a_{k',l'}(\omega)=0$.
On the other hand, for $l<l'$, k,k' arbitrary, observe that $a_{k,l}$ is $\mathcal{G}_{\sigma_l}$-measurable, i.e. also $\mathcal{G}_{\sigma_{l'-1}}$-measurable.
Since $E[M_{\sigma_{l'}} \mid \mathcal{G}_{\sigma_{l'-1}}] = M_{\sigma_{l'-1}}$ we see that $E[a_{k,l}a_{k',l'}]=0$, proving (4.47).

It follows that

$$E[(I_T^n - I_T^m)^2] = \sum_{k,l} E[a_{k,l}^2]$$

$$\leq E\Big[\sup_{k,l} |M_{\sigma_{l-1}} - M_{\rho_{k-1}}|^2 1\!\!1\{\rho_{k-1}\leq \sigma_{l-1}\leq \rho_k\}\cdot$$

(4.48) $$\sum_{k,l} 1\!\!1\{\rho_{k-1}\leq \sigma_{l-1}<\rho_k\}\,|M_{\sigma_l}-M_{\sigma_{l-1}}|^2\Big]$$

$$\leq E\Big[\sup_{k,l}|M_{\sigma_{l-1}}-M_{\rho_{k-1}}|^4 1\!\!1\{\rho_{k-1}\leq\sigma_{l-1}<\rho_k\}\Big]^{1/2}$$

$$E\Big[\Big(\sum_l |M_{\sigma_l}-M_{\sigma_{l-1}}|^2\Big)^2\Big]^{1/2}.$$

As M is bdd and continuous, the first term converges to 0 as $m,n\to\infty$, by DCT. For the second term,

$$\Big(\sum_l (M_{\sigma_l}-M_{\sigma_{l-1}})^2\Big)^2 = \sum_l (M_{\sigma_l}-M_{\sigma_{l-1}})^4 + 2\sum_{l<l'}(M_{\sigma_l}-M_{\sigma_{l-1}})^2(M_{\sigma_{l'}}-M_{\sigma_{l'-1}})^2$$

Observe, that $E\Big[\sum_{l'=l+1}^{k_m}(M_{\sigma_{l'}}-M_{\sigma_{l'-1}})^2 \mid \mathcal{G}_{\sigma_l}\Big] = E[(M_T-M_{\sigma_l})^2\mid\mathcal{G}_{\sigma_l}]$,
so

$$E\Big[\Big(\sum_l (M_{\sigma_l}-M_{\sigma_{l-1}})^2\Big)^2\Big] = E\Big[\sum_l (M_{\sigma_l}-M_{\sigma_{l-1}})^4\Big]$$
$$+\,2\sum_l E\big[(M_T-M_{\sigma_l})^2(M_{\sigma_l}-M_{\sigma_{l-1}})^2\big]$$

$$\leq E\Big[\Big(\sup_l (M_{\sigma_l}-M_{\sigma_{l-1}})^2 + 2\sup_l (M_T-M_{\sigma_l})^2\Big)\sum_l (M_{\sigma_l}-M_{\sigma_{l-1}})^2\Big]$$

Let C be such that $|M|\leq C$. Then

$$\leq 12C^2 E\Big[\sum_l (M_{\sigma_l}-M_{\sigma_{l-1}})^2\Big] = 12C^2 E[M_T^2] = 12C^4.$$

I.e. the second term in (4.48) is bounded, proving (4.45).

As I^n is martingale, the Doob's inequality (4.13) and (4.45) yield
$$\lim_{m,n\to\infty} E\Big[\sup_{t\le T}(I^n_t - I^m_t)^2\Big] = 0.$$
In particular, there is a subsequence n_k such that I^{n_k} converges uniformly a.s. on $[0,T]$ to a process $(I_t)_{t\le T}$ with continuous trajectories.

Moreover, for every $t<T$, $k\in\mathbb{N}$
$$E\big[(I^{n_k}_t)^2\big] \le C\,E\Big[\sum_i (M_{t^{n_k}_{i}\wedge t} - M_{t^{n_k}_{i-1}\wedge t})^2\Big] \le C\,E[M_t^2] \le C^3,$$
so the family $(I^{n_k}_t)_k$ is UI for every $t\le T$. Hence also

(4.49) $I^{n_k}_t \xrightarrow{k\to\infty} I_t$ in $L^1(P)$.

Using (4.49) we deduce that $(I_t)_{t\le T}$ is a continuous martingale. Moreover, from (4.43) we see that $M_t^2 - 2I^n_t$ is increasing along the partition t^n_i, so $M_t^2-2I_t$ is increasing on $[0,T]$, P-a.s. We thus set $\langle M\rangle_t = M_t^2 - 2I_t$ (and $\langle M\rangle \equiv 0$ on the null set where the limit does not exist). Then $\langle M\rangle$ is increasing and $M_t^2 - \langle M\rangle_t = 2I_t$ is a mart. on $[0,T]$.

To extend the definition to $t\in\mathbb{R}_+$, consider the same construction for $T=k$ and $T=k+1$, $k\in\mathbb{N}$, and use the essential uniqueness.

The uniqueness also shows that $\langle M\rangle$ does not depend on the sequence of partitions that was used for its construction and for any refining partitions
$$\lim_{n\to\infty}\sum_{i=1}^{k_n}(M_{t^n_i} - M_{t^n_{i-1}})^2 = \langle M\rangle_T \quad \text{in } L^2.$$

Case 2: M general local mart.

Write $M_t = M_0 + N_t$. Then $M_t^2 = M_0^2 + 2N_tM_0 + N_t^2$. Observing that $2N_tM_0$ is a local mart., we see that it is sufficient to consider the case $M_0=0$.

$$\text{Let } T_n = \inf\{t\ge 0: |M_t|\ge n\}.$$

Then T_n reduces M and M^{T_n} is bounded mrtg. Using the case 1, there exists $\langle M^{T_n}\rangle$ for every n. Due to essential uniqueness we also see that processes $\langle M^{T_{n+1}}\rangle_{t\wedge T_n}$ and $\langle M^{T_n}\rangle_{t\wedge T_n}$ are indistinguishable. So there is essentially unique process $\langle M\rangle_t$ such that $\langle M\rangle_{t\wedge T_n} = \langle M^{T_n}\rangle_{t\wedge T_n}$. By construction, $M^2_{t\wedge T_n} - \langle M\rangle_{t\wedge T_n}$ is a martingale (in L^2), so $M_t^2 - \langle M\rangle_t$ is a local mrtg.

The second part of the theorem then follows by replacing $M, \langle M\rangle$ with $M^{T_n}, \langle M^{T_n}\rangle$ for which the convergence holds in L^2 and observing that $P[T<T_n]\nearrow 1$ as $n\to\infty$ for every finite T $\square$.

Properties of quadratic variation

(4.50) **Lemma:** Let M be a local mtg and T a stopping time. Then
$$\langle M^T\rangle_t = \langle M\rangle_{t\wedge T} = \langle M\rangle^T_t .$$

Proof: Observe that $M^2_{t\wedge T} - \langle M\rangle_{t\wedge T}$ is a local mtg, by (4.37(iii)). I.e. $\langle M\rangle_{t\wedge T}$ satisfies the defining property of quadratic var. of M^T. $\square$

For an increasing process A_t we define $A_\infty = \lim_{t\to\infty} A_t$. The following theorem links the properties of $\langle M\rangle$ to properties of M.

(4.51) **Theorem:** Let M be a local mtg. with $M_0 = 0$
(i) If M is a mtg bounded in L^2, then $E[\langle M\rangle_\infty] < \infty$ and $M_t^2 - \langle M\rangle_t$ is a uniformly integrable martingale
(ii) If $E[\langle M\rangle_\infty] < \infty$, then M is a martingale bounded in L^2.
(iii) There is equivalence between
  (a) M is mtg with $E[M_t^2] < \infty$ $\forall t \in \mathbb{R}_+$
  (b) $E[\langle M\rangle_t] < \infty$ $\forall t \in \mathbb{R}_+$.
In this case $M_t^2 - \langle M\rangle_t$ is a mtg.

(4.52) **Remark:** There are L^2 bounded local mtgs which are not true martingales (we will see one later), so in (i) one should really assume that M is martingale. On technical side, the Doob's inequality does not hold for local mtgs.

Proof (i) If M is L^2 bounded martingale, the Doob inequality yield
$$E\Big[\sup_{t\geq 0} M_t^2\Big] \leq 4 \sup_{t\geq 0} E[M_t^2] < \infty$$
Hence M is uniformly integrable. By (4.21), there is $M_\infty = \lim_t M_t$ a.s. & in L^2.

For $n \in \mathbb{N}$, let $S_n = \inf\{t \geq 0 : \langle M \rangle_t \geq n\}$. Then S_n is a stopping time and $\langle M \rangle_{t \wedge S_n} \leq n$. Hence, the local mtg $M^2_{t \wedge S_n} - \langle M \rangle_{t \wedge S_n}$ is dominated by the integrable random variable $\sup_{t \geq 0} M_t^2 + n$. Hence, by (4.39), it is a true mtg, and thus

$$E[\langle M \rangle_{t \wedge S_n}] = E[M^2_{t \wedge S_n}].$$

Letting $t \to \infty$, using MCT on the LHS and DCT on the RHS, yields

$$E[\langle M \rangle_{S_n}] = E[\langle M^2 \rangle_{S_n}].$$

Using the same arguments for $n \to \infty$, we then find

$$E[\langle M \rangle_\infty] = E[M^2_\infty] < \infty$$

In particular, the local mtg $M_t^2 - \langle M \rangle_t$ is dominated by integrable random variable $\sup_t M_t^2 + \langle M \rangle_\infty$, so it is UI mtg, by (4.39) again.

– (ii): Assume that $E[\langle M \rangle_\infty] < \infty$. Let $T_n = \inf\{t : |M_t| \geq n\}$. i.e. M^{T_n} is a bounded mtg. The local mtg $M^2_{t \wedge T_n} - \langle M \rangle_{t \wedge T_n}$ is dominated by $n + \langle M \rangle_\infty$, so it is UI mtg by (4.39). Using the stopping theorem, (4.22), for a stopping time S which is a.s. finite

$$E[M^2_{S \wedge T_n}] = E[\langle M \rangle_{S \wedge T_n}]$$

and thus by Fatou's lemma and MCT

(4.53) $$E[M_S^2] \leq E[\langle M \rangle_S] \leq E[\langle M \rangle_\infty] < \infty.$$

Hence, the family

$$\{M_S : S \text{ is a finite stopping time}\} \text{ is UI.}$$

We may thus pass to the L^1-limit as $n \to \infty$ in the equality $E[M_{t \wedge T_n} \mid \mathcal{F}_s] = M_{s \wedge T_n}$, $s \leq t$, to deduce $E[M_t \mid \mathcal{F}_s] = M_s$, i.e. M is a mtg. It is bdd in L^2 by (4.53)

– (iii) Let $a > 0$. By (i), (ii) $E[\langle M \rangle_a] < \infty$ iff M_t^a is L^2-bounded true mtg. This proves the equivalence between (a), (b). (i) then implies that $M^2_{t \wedge a} - \langle M \rangle_{t \wedge a}$ is a mtg and the last claim follows. $\square$

(4.54) Corollary: Let M be a local mtg with $M_0 = 0$. Then $\langle M, M\rangle_t = 0$ a.s. for all $t \geq 0$ iff M is indistinguishable from 0.

Proof: If $\langle M, M\rangle_t = 0$ a.s. for all $t \geq 0$, then by (ii) and (i) of the above theorem, M_t^2 is a UI mtg. I.e $E[M_t^2] = E[M_0^2] = 0$. Opposite implication is obvious. $\square$.

Bracket process of two local mtgs.

(4.55) Definition. If M and N are two local martingales we define

$$\langle M, N\rangle_t = \frac{1}{2}\left(\langle M+N\rangle_t - \langle M\rangle_t - \langle N\rangle_t\right)$$

(4.56) Remark: With this notation, $\langle M, M\rangle_t = \langle M\rangle_t$. Some authors thus prefer to write $\langle M, M\rangle$ instead of $\langle M\rangle$.

(4.57) Proposition: (a) $\langle M, N\rangle$ is the (essentially) unique process of finite variation started from 0 such that $M_t N_t - \langle M, N\rangle_t$ is a local martingale

(b) The map $(M, N) \mapsto \langle M, N\rangle$ is bilinear and symmetric.

(c) If $0 = t_0^n < \ldots < t_{k_n}^n = t$ is a sequence of refining partitions of $[0, t]$ with meshsize tending to 0, then

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} (M_{t_i^n} - M_{t_{i-1}^n})(N_{t_i^n} - N_{t_{i-1}^n}) = \langle M, N\rangle_t \text{ in probability.}$$

(d) For every stopping time T

$$\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M, N\rangle_{t\wedge T}.$$

Proof: - (a) uniqueness follows as in Theorem (4.41) and the stated property follows directly from the analogous statement in (4.41).
- (c) follows from the analogous statement in (4.41) as well
- (b) is consequence of (c)

- (d) From (c) we can see that a.s.

$$\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M, N\rangle_t \quad \text{on } \{T \geq t\}$$

$$\langle M^T, N^T\rangle_t - \langle M^T, N^T\rangle_s = \langle M^T, N\rangle_t - \langle M^T, N\rangle_s = 0 \quad \text{on } \{T \leq s < t\}$$

which yields (d). □

(4.58) Definition: Two local mtgs M, N are called orthogonal if $\langle M, N\rangle = 0$, in which case MN is a local mtg.

(4.59) Example. Let B be a $\mathcal{G}_t$-Brownian motion (i.e. it is $\mathcal{G}_t$-adapted and $B_t - B_s$ is independent of $\mathcal{G}_s$, $s \leq t$). Then B is a (local) martingale and $\langle B\rangle_t = t$. If B' is another $\mathcal{G}_t$-Brownian motion, independent of B, then B and B' are orthogonal. To see this observe that $\frac{1}{\sqrt{2}}(B_t + B'_t)$ is again a BM.

(4.60) Proposition (Kunita-Watanabe inequality).
Let M, N be two local mtgs and H, K two measurable processes. Then

$$\int_0^\infty |H_s||K_s|\,|d\langle M, N\rangle_s| \leq \left(\int_0^\infty H_s^2\, d\langle M\rangle_s\right)^{1/2}\left(\int_0^\infty K_s^2\, d\langle N\rangle_s\right)^{1/2} \quad P\text{-a.s.}$$

(here $|d\langle M, N\rangle_s|$ denotes the total variation of the signed measure $d\langle M, N\rangle_s$.)

Proof: For $s < t$, define $\langle M, N\rangle_s^t = \langle M, N\rangle_t - \langle M, N\rangle_s$. Then P-a.s. for all $\lambda \in \mathbb{Q}$, $s < t \in \mathbb{Q}$, $\langle M + \lambda N\rangle_s^t = \langle M\rangle_s^t + 2\lambda\langle M, N\rangle_s^t + \lambda^2\langle N\rangle_s^t \geq 0$.
Taking discriminant in λ, we deduce that P-a.s. $\forall s, t \in \mathbb{Q}$

(4.61) $$|\langle M, N\rangle_s^t| \leq \left(\langle M\rangle_s^t \langle N\rangle_s^t\right)^{1/2}$$

By continuity this extends to all $s \leq t \in \mathbb{R}_+$. From this we get

(4.62) $$\int_s^t |d\langle M, N\rangle_u| \leq \left(\langle M\rangle_s^t \langle N\rangle_s^t\right)^{1/2} = \left(\int_s^t d\langle M\rangle_u \int_s^t d\langle N\rangle_u\right)^{1/2}$$

Indeed, for every subdivision $s = t_0 < \dots < t_n = t$,

$$\sum_{i=1}^n |\langle M, N\rangle_{t_{i-1}}^{t_i}| \overset{(4.61)}{\leq} \sum_{i=1}^n \left(\langle M\rangle_{t_{i-1}}^{t_i} \langle N\rangle_{t_{i-1}}^{t_i}\right)^{1/2} \overset{CS}{\leq} \left(\sum_{i=1}^n \langle M\rangle_{t_{i-1}}^{t_i}\right)^{1/2}\left(\sum_{i=1}^n \langle N\rangle_{t_{i-1}}^{t_i}\right)^{1/2}$$

$$\leq \left(\langle M\rangle_s^t \langle N\rangle_s^t\right)^{1/2},$$

and (4.62) follows from properties of L-S. integral, e.g. (4.29).

(4.62) implies that for every Borel set $A \subset \mathbb{R}_+$, by similar arguments,

(4.63) $$\int_A |d\langle M,N\rangle_u| \le \left(\int_A d\langle M\rangle_u \int_A d\langle N\rangle_u \right)^{1/2}.$$

Let now $h = \sum \lambda_i \mathbb{1}_{A_i}$, $k = \sum \mu_i \mathbb{1}_{A_i}$ be two positive step functions, then by CS inequality again,

$$\int h(s)k(s)\,|d\langle M,N\rangle_s| = \sum_i \lambda_i \mu_i \int_{A_i} |d\langle M,N\rangle_s|$$
$$\le \left(\sum_i \lambda_i^2 \int_{A_i} d\langle M\rangle_s \right)^{1/2} \left(\sum_i \mu_i^2 \int_{A_i} d\langle N\rangle_s \right)^{1/2}$$
$$= \left(\int h^2(s)\, d\langle M\rangle_s \right)^{1/2} \left(\int k^2(s)\, d\langle N\rangle_s \right)$$

for a.e. ω where (4.63) holds. It remains to write every measurable $H_s(\omega)$, $K_s(\omega)$ as difference of two limits of sequences of positive step functions $\square$

4.9 Continuous semimartingales

(4.64) <u>Definition:</u> A process $(X_t)_{t\geq 0}$ is called <u>continuous semimartingale</u> if it can be written as

$$X_t = X_0 + M_t + A_t$$

where M_t is a local mtg (with $M_0 = 0$) and A is process of finite variation.

The above decomposition is essentially unique. This follows from (4.40) using the standard arguments.

If $Y = Y_0 + N_t + B_t$ is another continuous semimtg with the corresponding decomposition, we define

$$\langle X, Y \rangle_t = \langle M, N \rangle_t$$

In particular, $\langle X \rangle_t = \langle M \rangle_t$.

(4.65) <u>Proposition:</u> Let $(t_i^n)_{i=1,\dots,k_n}$ be a refining sequence of partitions of $[0,t]$ with mesh tending to 0. Then

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} (X_{t_i^n} - X_{t_{i-1}^n})(Y_{t_i^n} - Y_{t_{i-1}^n}) = \langle X, Y \rangle_t, \text{ in probability.}$$

<u>Proof:</u> We consider only $X = Y$, general result follows by polarisation.

$$\sum_{i=1}^{k_n} (X_{t_i^n} - X_{t_{i-1}^n})^2 = \sum_i (M_{t_i^n} - M_{t_{i-1}^n})^2 + \sum_i (A_{t_i^n} - A_{t_{i-1}^n})^2 + 2\sum_i (M_{t_i^n} - M_{t_{i-1}^n})(A_{t_i^n} - A_{t_{i-1}^n})$$

The first term converges to $\langle M \rangle_t$ by (4.41), and $\langle M \rangle = \langle X \rangle$.

On the other hand

$$\sum_i (A_{t_i^n} - A_{t_{i-1}^n})^2 \leq \Big(\underbrace{\sup_i |A_{t_i^n} - A_{t_{i-1}^n}|}_{\xrightarrow{n\to\infty} 0}\Big) \underbrace{\sum |A_{t_i^n} - A_{t_{i-1}^n}|}_{\leq \int_0^t |dA_s| < \infty} \xrightarrow[n\to\infty]{MCT} 0.$$

Same arguments prove also that the last term tends to 0 as $n\to\infty$. □

Chapter V. STOCHASTIC INTEGRAL

We now develop the theory of the stochastic integral with respect to martingales, and later semimartingales. We know that these processes are a.s. of infinite variation, which precludes the definition of "Stieltjes-type" integral w.r.t. $dM_t(\omega)$ pathwise. We will see that the key role will be played by the quadratic variation process $\langle M \rangle$ constructed in the last chapter.

The stochastic integral was first constructed by Itô (1942) for BM, and extended to general martingales by Kunita & Watanabe ('67).

In the whole chapter we assume that $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ satisfies the usual conditions.

(5.1) <u>Definition</u>: We denote by H^2 the space of continuous martingales M which are bounded in L^2, $\sup_t \|M_t\|^2 < \infty$, with $M_0 = 0$.

By (4.51), if $M \in H^2$, then $E\langle M\rangle_\infty < \infty$, and thus if $M, N \in H^2$, then $E|\langle M, N\rangle_\infty| < \infty$. By Kunita-Watanabe inequality (4.60)

(5.2) $$E|\langle M,N\rangle_\infty| \leq E\Big[\int_0^\infty |d\langle M,N\rangle_s|\Big] \leq E[\langle M\rangle_\infty]^{1/2} E[\langle N\rangle_\infty]^{1/2}.$$

We can thus define a scalar product on H by

(5.3) $$(M,N)_{H^2} = E[\langle M,N\rangle_\infty]$$

By (4.54), $(M,M)_{H^2} = 0$ iff $M \equiv 0$ (identifying the indistinguishable processes). We also define a norm on H^2

(5.4) $$\|M\|_{H^2} = (M,M)_{H^2}^{1/2} = E[\langle M\rangle_\infty]^{1/2}.$$

(5.5) <u>Proposition</u>: H^2 endowed with the scalar product $(\cdot,\cdot)_{H^2}$ is a Hilbert space.

<u>Proof</u>: We need to show that H^2 is complete. Let M^n be a Cauchy sequence in H^2. Then, by definition of $\langle\cdot\rangle$,

$$\lim_{m,n\to\infty} E\left[(M^n_\infty - M^m_\infty)^2\right] = \lim_{m,n} E\left[\langle M^n - M^m\rangle_\infty\right] = 0.$$

Thus, by Doob's inequality,

(5.6) $$\lim_{n,m\to\infty} E\left[\sup_{t\geq 0} (M^n_t - M^m_t)^2\right] = 0.$$

Hence, along some subsequence (n_k),

(5.7) $$\lim_{l,k\to\infty} \sup_{t\geq 0} |M^{n_k}_t - M^{n_l}_t| = 0, \quad P\text{-a.s.}$$

Therefore, P-a.s, there exists a limit $M_t(\omega) = \lim_k M^{n_k}_t(\omega)$, and by the uniform convergence (5.7) this limit is continuous in t. Setting $M \equiv 0$ on the P-negligible set where M is not defined, we see from (5.6) that $M^{n_k}_t \to M_t$ also in L^2. Passing to the limit in the martingale property of M^{n_k}, we see that M is a mtg. Moreover, since M^n_t are uniformly bounded in L^2, the same is true for M_t. Finally

$$\lim_k E\left[\langle M^{n_k} - M\rangle_\infty\right] = \lim_k E\left[(M^{n_k}_\infty - M_\infty)^2\right] = 0,$$

so M^{n_k} converges to M also in H^2 $\square$.

Recall that a process H is prog. measurable if the map

$$([0,t]\times\Omega, \mathcal{B}([0,t])\otimes\mathcal{G}_t) \ni (t,\omega) \mapsto H_t(\omega) \in (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable for all $t\geq 0$, and that $A \in \mathcal{B}(\mathbb{R}_+)\otimes\mathcal{G}$ is called prog. measurable if $(t,\omega)\mapsto 1_A(t,\omega)$ is prog. measurable process. We define <u>progressive σ-algebra Prog</u> as the smallest σ-algebra containing all prog. measurable sets.

(5.8) <u>Exercise</u>: H is prog. measurable iff $(\mathbb{R}_+\times\Omega, \text{Prog}) \ni (t,\omega) \mapsto H_t(\omega) \in (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable.

For $M \in H^2$, we define

(5.9) $$L^2(M) = L^2(\mathbb{R}_+ \times \Omega, \text{Prog}, dP\, d\langle M\rangle_s)$$

That is $L^2(M)$ is the space of prog. measurable processes H such that

$$E\left[\int_0^\infty H_s^2 \, d\langle M\rangle_s\right] < \infty.$$

As every L^2-space, $L^2(M)$ is a Hilbert space with scalar product

$$(H,K)_{L^2(M)} = E\left[\int_0^\infty H_s K_s \, d\langle M\rangle_s\right]$$

(5.10) Definition: A process H is called simple if

$$H_s(\omega) = \sum_{i=0}^{n-1} H_{(i)}(\omega)\, 1\!\!1_{(t_i, t_{i+1}]}(s),$$

where $n \in \mathbb{N}$, $0 = t_0 < \dots < t_n$, and $H_{(i)}$ is a bounded $\mathcal{G}_{t_i}$-meas. r.v.
We use $\mathcal{E}$ to denote the vector space of all simple processes.

(5.11) Proposition: For every $M \in H^2$, $\mathcal{E}$ is dense in $L^2(M)$.

Proof: It is sufficient to show that if $K \in L^2(M)$ is orthogonal to $\mathcal{E}$, then $K = 0$. Assume that K is orthogonal to $\mathcal{E}$. Let $0 \le s < t$, and let Z be $\mathcal{G}_s$-measurable bdd r.v. Then, by orthogonality,

(5.12) $$0 = (K, Z 1\!\!1_{(s,t]}) = E\left[Z \int_s^t K_u \, d\langle M\rangle_u\right].$$

Let now,

(5.13) $$X_t = \int_0^t K_u \, d\langle M\rangle_u, \quad t \ge 0.$$

By CS inequality $X_t \le \left(\int_0^t K_u^2 d\langle M\rangle_u\right)^{1/2} \left(\int_0^t d\langle M\rangle_u\right)^{1/2} < \infty$ a.s. as $K \in L^2(M)$ and $M \in H^2$, so X_t is a.s. well defined, and also that $X_t \in L^1$. (5.12) implies that $E[Z(X_t - X_s)] = 0$, so X is a mtg. By (5.13), $X_0 = 0$ and X has finite variation. So, by (4.26), $X = 0$. Hence

$$\int_0^t K_u \, d\langle M\rangle_u = 0 \quad \forall t \ge 0, \text{ a.s.}$$

so

$$K_u = 0 \quad d\langle M\rangle\text{-a.e. } P\text{-a.s.}$$

so $K = 0$ in $L^2(M)$. □

(5.14) <u>Theorem:</u> Let $M \in H^2$. For $H \in \mathcal{E}$ of the form

$$H_s(\omega) = \sum_{i=0}^{n-1} H_{(i)}(\omega)\, 1\!\!1_{(t_i, t_{i+1}]}(s)$$

we define $H \cdot M \in H^2$ by

(5.15) $$(H \cdot M)_t = \sum_{i=0}^{n-1} H_{(i)}(\omega) \left(M_{t_{i+1} \wedge t} - M_{t_i \wedge t} \right)$$

The map $H \mapsto H \cdot M$ can be extended to an isometry of $L^2(M)$ into H^2. Moreover $H \cdot M$ is characterised by the relation

(5.16) $$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle \quad \text{for all } N \in H^2,$$

and if T is a stopping time, then

(5.17) $$(1\!\!1_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T$$

(5.18) <u>Notation</u> $H \cdot M$ is called the stochastic integral of H w.r.t. M and is often denoted as

$$(H \cdot M)_t = \int_0^t H_s \, dM_s.$$

<u>Remarks:</u>
- The formula (5.16) is called martingale characterisation of the stoch. integral and is used to define it in some books. Observe that the integral on the RHS of (5.16) is w.r.t. a process of finite variation that we know well already.
- The isometry of Theorem (5.14) is called Itô's isometry.

<u>Proof:</u> <u>Step 1</u>: $\mathcal{E} \ni H \mapsto H \cdot M$ is isometry from $L^2(M)$ into H^2.
We define $M_t^i = H_{(i)} \left(M_{t_{i+1} \wedge t} - M_{t_i \wedge t} \right)$, so $H \cdot M_t = \sum_{i=0}^{n-1} M_t^i$.
It is trivial to verify that M^i is a mtg which is continuous and bounded in L^2, so $M^i \in H^2$ and thus $H \cdot M \in H^2$.
The linearity of the map $H \mapsto H \cdot M$ is obvious. Moreover, M^i, $i = 0, \dots, n-1$ are orthogonal (i.e. $\langle M^i, M^j \rangle = 0$ if $i \neq j$) and

$$\langle M^i \rangle_t = H_{(i)}^2 \left(\langle M \rangle_{t \wedge t_{i+1}} - \langle M \rangle_{t \wedge t_i} \right)$$

Hence

$$\langle H\cdot M\rangle_t = \sum_{i=0}^{n-1} H_{(i)}^2 \left(\langle M\rangle_{t\wedge t_{i+1}} - \langle M\rangle_{t\wedge t_i}\right)$$

and thus

$$\|H\cdot M\|_{H^2}^2 = E\left[\sum_{i=0}^{n-1} H_{(i)}^2\left(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}\right)\right]$$

$$= E\left[\int_0^t H_s^2\, d\langle M\rangle_s\right] = \|H\|^2_{L^2(M)}$$

proving the isometry property.

Step 2: Existence of an extension.

By Step 1, $H \mapsto H\cdot M$ is isometry from $\mathcal{E}$ into H^2. As $\mathcal{E}$ is dense in $L^2(M)$, this map extends uniquely to an isometry of $L^2(M)$ into H^2, as claimed.

Step 3: $H\cdot M$ satisfies (5.16)

Let $H\in\mathcal{E}$ as above. Then $\langle H\cdot M, N\rangle = \sum_{i=0}^{n-1}\langle M^i, N\rangle$.

Moreover, $\langle M^i, N\rangle_t = H_{(i)}\left(\langle M,N\rangle_{t\wedge t_{i+1}} - \langle M,N\rangle_{t\wedge t_i}\right)$

Hence

$$\langle H\cdot M, N\rangle_t = \sum_{i=0}^{n-1} H_{(i)}\left(\langle M,N\rangle_{t\wedge t_{i+1}} - \langle M,N\rangle_{t\wedge t_i}\right) = \int_0^t H_s\, d\langle M,N\rangle_s$$

proving (5.16) for $H\in\mathcal{E}$.

We now claim that the map $X\mapsto \langle X,N\rangle_\infty$ is continuous from H^2 to L^1. Indeed

$$E\left[|\langle X,N\rangle_\infty|\right] \le E\left[\langle X,X\rangle_\infty\right]^{1/2} E\left[\langle N,N\rangle_\infty\right]^{1/2} = \|X\|_{H^2}\,\|N\|_{H^2},$$

proving the continuity. Hence, if $H^n\in\mathcal{E}$, $H^n\to H$ in $L^2(M)$, then

(5.19) $$\langle H\cdot M, N\rangle_\infty \overset{\text{def}}{=} \lim_{n\to\infty}\langle H^n\cdot M, N\rangle_\infty \underset{\text{on }\mathcal{E}}{\overset{(5.16)}{=}} \lim\left(H^n\cdot\langle M,N\rangle\right)_\infty$$

$$= \left(H\cdot\langle M,N\rangle\right)_\infty,$$

where the last equality follows by Kunita–Watanabe inequality:

$$E\left[\left|\int_0^\infty (H^n - H)\, d\langle M,N\rangle_s\right|\right] \le E\left[\langle N,N\rangle_\infty\right]^{1/2}\|H-H^n\|_{L^2(M)}.$$

Replacing N by N^t in (5.19), using (4.57(d)), we find (5.16) for $H\in L^2(M)$ arbitrary.

Step 4. (5.16) determines $H \cdot M$.

Assume that $X \in H^2$ satisfying $\langle X, N \rangle_t = \int_0^t H_s d\langle M, N \rangle_s \; \forall N \in H^2$

Then $\langle H \cdot M - X, N \rangle = 0$ and in particular taking $N = H \cdot M - X$

$\langle H \cdot M - X \rangle = 0$. (4.54) then yields $X = H \cdot M$.

Step 5 Proof of (5.17). Using (4.57(d)), for $N \in H^2$

$$\langle (H \cdot M)^T, N \rangle_t = \langle (H \cdot M), N \rangle_{t \wedge T} \overset{(5.16)}{=} (H \cdot \langle M, N \rangle)_{t \wedge T} =$$
$$= (H 1\!\!1_{[0,T]} \cdot \langle M, N \rangle)_t = \langle (H 1\!\!1_{[0,T]}) \cdot M, N \rangle$$

and thus by (5.16) $(H \cdot M)^T = (H 1\!\!1_{[0,T]}) \cdot M$. For the second equality we write

$$\langle H \cdot M^T, N \rangle = H \cdot \langle M^T, N \rangle = H \cdot \langle M, N \rangle^T = H 1\!\!1_{[0,T]} \cdot \langle M, N \rangle$$

completing the proof $\square$

(5.20) <u>Proposition</u>: If $K \in L^2(M)$ and $H \in L^2(K\cdot M)$, then $HK \in L^2(M)$ and

$$(HK)\cdot M = H\cdot(K\cdot M)$$

<u>Proof</u>: By (5.16),

$$\langle K\cdot M, K\cdot M\rangle_s = K\cdot\langle M, K\cdot M\rangle = K^2\cdot\langle M,M\rangle,$$

and thus

$$\int_0^\infty H_s^2 K_s^2 \, d\langle M\rangle_s = \int_0^\infty H_s^2 \, d\langle K\cdot M\rangle_s < \infty$$

implying the first claim. For the second, observe that for $N \in H^2$

$$\langle (HK)\cdot M, N\rangle = HK\cdot\langle M,N\rangle = H\cdot(K\cdot\langle M,N\rangle) = H\cdot\langle K\cdot M, N\rangle =$$
$$= \langle H\cdot(K\cdot M), N\rangle$$ □.

(5.21) <u>Remark</u>: "Informally", we may write (5.20) as

$$\int_0^t H_s (K_s dM_s) = \int_0^t H_s K_s \, dM_s$$

Similarly, (5.16) can be written as

$$\left\langle \int_0^\cdot H_s dM_s, N\right\rangle_t = \int_0^t H_s \, d\langle M,N\rangle_s$$

and thus also

$$\left\langle \int_0^\cdot H_s dM_s, \int_0^\cdot K_s dN_s\right\rangle_t = \int_0^t H_s K_s \, d\langle M,N\rangle_s$$

in particular

$$\left\langle \int_0^\cdot H_s dM_s\right\rangle_t = \int_0^t H_s^2 \, d\langle M\rangle_s .$$

Finally, since if $M, N \in H^2$, $H \in L^2(M)$, $K \in L^2(N)$, then $H\cdot M, K\cdot N \in H^2$, so

$$E\left[\int_s^t H_u dM_u \,\middle|\, \mathcal{G}_s\right] = 0$$

$$E\left[\left(\int_s^t H_u dM_u\right)\left(\int_s^t K_u dN_u\right) \,\middle|\, \mathcal{G}_s\right] = E\left[\int_s^t H_s K_s \, d\langle M,N\rangle_s \,\middle|\, \mathcal{G}_s\right].$$

This relations will not necessary valid for extensions of the stoch integral given below.

5.2 <u>Stochastic integral wrt. local mrtgs</u>

We now extend the definition of $H\cdot M$ to continuous local mrtgs. Let M be a (cont.) local martingale with $M_0 = 0$. We define

$$L^2_{loc}(M) = \left\{ H \text{ prog. process} : \forall t \geq 0 \ \int_0^t H_s^2 \, d\langle M\rangle_s < \infty \text{ a.s} \right\}$$

$$L^2(M) = \left\{ H \text{ prog. process} : E\left[\int_0^\infty H_s^2 \, d\langle M\rangle_s\right] < \infty \right\}.$$

(5.22) Theorem: Let M be a local mtg. with $M_0=0$. Then for every $H\in L^2_{loc}(M)$ there is an ess. unique local martingale started from 0, denoted $H\cdot M$, s.t. for every local martingale N

(5.23) $$\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$$

The property (5.17) holds, and if $M\in H^2$, $H\in L^2(M)$, then this definition coincides with the one of Thm (5.14).

Proof: Let
$$T_n = \inf\left\{t\geq 0:\ \int_0^t (1+H_s^2)\,d\langle M\rangle_s \geq n\right\}.$$
T_n is an increasing sequence of st. times tending to ∞ with n. Since $\langle M^{T_n}\rangle_t = \langle M\rangle_{T_n\wedge t}\leq n$, it follows from (4.51(ii)) that $M^{T_n}\in H^2$. Also, $\int_0^\infty H_s^2\, d\langle M^{T_n}\rangle_s \leq n$, that is $H\in L^2(M^{T_n})$. Hence, for every n, the stoch. integral $H\cdot M^{T_n}$ is well defined. Using (5.16), (5.17), one sees that for $m>n$,
$$H\cdot M^{T_n} = (H\cdot M^{T_m})^{T_n}$$
This implies that there exists unique process $H\cdot M$ such that
$$(H\cdot M)^{T_n} = H\cdot M^{T_n} \qquad \forall n\in\mathbb{N}.$$
Therefore also $(H\cdot M)^{T_n}\in H^2$ and thus $H\cdot M$ is a local mtg.

Let N be a local mtg (w.l.o.g. $N_0=0$) and set $T_n' = \inf\{t\geq 0: |N_t|\geq n\}$, $S_n = T_n\wedge T_n'$. Then
$$\langle H\cdot M, N\rangle^{S_n} = \langle (H\cdot M)^{S_n}, N^{S_n}\rangle = \langle (H\cdot M^{T_n})^{S_n}, N^{S_n}\rangle$$
$$= \langle H\cdot M^{T_n}, N^{S_n}\rangle \overset{(5.16)}{=} H\cdot\langle M^{T_n}, N^{S_n}\rangle = H\cdot\langle M,N\rangle^{S_n}$$
$$= (H\cdot\langle M,N\rangle)^{S_n}$$
and thus $\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$. The fact that this equality characterises $H\cdot M$ is proved as in Theorem (5.14).

(5.17) is proved as before, as its proof only uses (5.16) which we already extended. Finally, for $M\in H^2$, $H\in L^2(M)$, (5.23) shows $\langle H\cdot M\rangle = H^2\cdot\langle M\rangle$ that is $H\cdot M\in H^2$, so by (5.23), (5.16) the both definitions coincide □.

(5.24) <u>Remark</u>: Moment formulas of Remark (5.21) need an additional assumption: If M is a local mg and $H \in L^2_{loc}(M)$, $t \in \mathbb{R}_+ \cup \{\infty\}$, then if (!)

$$E[\langle H\cdot M\rangle_t] = E\left[\int_0^t H_s^2 \, d\langle M\rangle_s\right] < \infty,$$

then

$$E\left[\int_0^t H_s dM_s\right] = 0; \quad E\left[\left(\int_0^t H_s dM_s\right)^2\right] = E\left[\int_0^t H_s^2 d\langle M\rangle_s\right].$$

Finally, we extend the stoch. integral to semimartingales.
We say that a prog. process H is loc. bounded if

$$P\text{-a.s.} \quad \forall t \geq 0 \quad \sup_{s\leq t} |H_s| < \infty.$$

In particular, all adapted cont processes are loc. bdd.
If H is loc. bdd and A of finite variation, then

$$P\text{-a.s.} \quad \forall t \geq 0 \quad \int_0^t |H_s||dA_s| < \infty$$

and also, for every loc. mg M, $H \in L^2_{loc}(M)$

(5.25) <u>Definition</u>: Let $X = X_0 + M + A$ be a cont. semimartingale and H a loc. bdd prog. process. The stoch. integral $H\cdot X$ is defined by

$$H\cdot X = H\cdot M + H\cdot A$$

and denoted

$$(H\cdot X)_t = \int_0^t H_s dX_s.$$

(5.26) <u>Properties</u>: (Exercise!)

(i) $(H, X) \longmapsto H\cdot X$ is bilinear

(ii) $H\cdot(K\cdot X) = (HK)\cdot X$ if H, K are loc bdd

(iii) For every st. time T, $H 1\!\!1_{[0,T]}\cdot X = H\cdot X^T = (H\cdot X)^T$

(iv) If X is a local mg (or of finite variation), the same holds for $H\cdot X$

(v) If $H_s(\omega) = \sum_{i=0}^{n-1} H_{(i)}(\omega) 1\!\!1_{(t_i, t_{i+1}]}(s) \in \mathcal{E}$, then $(H\cdot X)_t = \sum_{i=0}^{n-1} H_{(i)}(X_{t_{i+1}\wedge t} - X_{t_i \wedge t})$

We close this section by an useful approximation property

(5.27) <u>Proposition:</u> Let X be a semimartingale and H adapted continuous. Then for every sequence $(t_i^n)_{i\le k_n}$ of refining partitions of $[0,t]$ with step tending to 0

$$\lim_{n\to\infty}\sum_{i=0}^{k_n-1} H_{t_i^n}\left(X_{t_{i+1}^n}-X_{t_i^n}\right) = (H\cdot X)_t \quad \text{in probability.}$$

<u>Proof:</u> Let $X = X_0 + M + A$, and set

$$H_s^n = \begin{cases} H_{t_i^n}^n & \text{if } t_i^n < s \le t_{i+1}^n \\ 0 & \text{if } s > t \end{cases}$$

For the part of finite variation, for P-a.e. ω,

$$\sum_{i=0}^{k_n-1} H_{t_i^n}(\omega)\left(A_{t_{i+1}^n}(\omega)-A_{t_i^n}(\omega)\right) = \int_0^t H_s^n(\omega)\,dA_s(\omega) \xrightarrow[n\to\infty]{DCT} \int_0^t H_s(\omega)\,dA_s(\omega)$$

For the local martingale part, define for $p\ge 1$

$$T_p = \inf\{s\ge 0 : |H_s| + \langle M\rangle_s \ge p\}.$$

Then H, H^n and $\langle M\rangle$ are bounded on $[0,T_p)$. By L^2 theory of the stoch. integral

$$\left\|(H^n 1\!\!1_{[0,T_p)}\cdot M^{T_p})_t - (H 1\!\!1_{[0,T_p)}\cdot M^{T_p})_t\right\|_2 = E\left[\int_0^{t\wedge T_p}(H_s^n - H_s)^2\,d\langle M\rangle_s\right]$$

which tends to 0 by DCT. Using (5.17) we deduce that

$$(H^n\cdot M)_{t\wedge T_p} \longrightarrow (H\cdot M)_{t\wedge T_p} \quad \text{in } L^2$$

As $P[T_p > t] \nearrow 1$ as $p\nearrow\infty$, the claim follows $\square$.

5.3. Itô's formula

Itô's formula is a principal tool of the stochastic analysis. It replaces the usual deterministic formula

(5.28) $$f(F(t)) = f(F(0)) + \int_0^t f'(F(s))\,dF(s)$$

which holds for $f \in C^1$ and F of finite variation. In differential form, (5.28) is just the chain rule $\frac{d}{dt} f(F(t)) = f'(F(t)) \frac{d}{dt} F(t)$.

Itô's formula also shows that a sufficiently smooth function of a semimartingale is again a semimartingale and provides its decomposition.

(5.29) Theorem [Itô's formula (Ito 1944, Kunita-Watanabe 1967)]

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a C^2 function and $X_1, \dots, X_k$ continuous semimgs. Then

$$f(X_t^1, \dots, X_t^k) = f(X_0^1, \dots, X_0^k) + \sum_{i=1}^{k} \int_0^t \frac{\partial f}{\partial x_i}(X_s^1, \dots, X_s^k)\,dX_s^i$$
$$+ \frac{1}{2} \sum_{i,j=1}^{k} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s^1, \dots, X_s^k)\,d\langle X^i, X^j\rangle_s.$$

(5.30) Remark: The third term on the RHS is of finite variation, the second can be then written as $\sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(X_s^1, \dots, X_s^k)\,dM_s^i + \sum_{i=1} \int_0^t \frac{\partial f}{\partial x_i}(X_s^1, \dots, X_s^k)\,dA_s^i$, where $X_t^i = X_0^i + M_t^i + A_t^i$ is the decomposition of the semimg X^i. From these two terms, the first is a local mg and the second of finite variation.

Proof: We consider first the case $k=1$ and write $X^1 = X$.

Let $0 = t_0^n < \dots < t_{k_n}^n = t$ be a refining sequence of partitions of $[0,t]$. Then

$$f(X_t) = f(X_0) + \sum_{i=0}^{k_n - 1} \left(f(X_{t_{i+1}^n}) - f(X_{t_i^n}) \right)$$

Using Taylor expansion

$$f(X_{t_{i+1}^n}) - f(X_{t_i^n}) = f'(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n}) + \frac{\xi_{n,i}(\omega)}{2}(X_{t_{i+1}^n} - X_{t_i^n})^2$$

where

(5.31) $\xi_{n,i}(\omega) = f''(s)$ for some $s \in [X_{t_i^n} \wedge X_{t_{i+1}^n}, X_{t_i^n} \vee X_{t_{i+1}^n}]$.

Using Proposition (5.27) with $H_s = f'(X_s)$ we see that

$$\lim_{n\to\infty} \sum_{i=0}^{k_n-1} f'(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n}) = \int_0^t f'(X_s)\,dX_s, \text{ in probability.}$$

Hence, to complete the proof we should show that

(5.32) $$\lim_{n\to\infty} \sum_{i=0}^{k_n-1} \xi_{n,i}(X_{t_{i+1}^n} - X_{t_i^n})^2 = \int_0^t f''(X_s)\,d\langle X\rangle_s, \text{ in probability.}$$

To show (5.32), observe that for $m<n$

(5.33) $$\Big| \sum_{i=0}^{k_n-1} \xi_{n,i}(X_{t_{i+1}^n} - X_{t_i^n})^2 - \sum_{j=0}^{k_m-1} \xi_{m,j} \sum_{i: t_j^m \le t_i^n < t_{j+1}^m} (X_{t_{i+1}^n} - X_{t_i^n})^2 \Big|$$
$$\le Z_{m,n}\Big(\sum_{i=0}^{k_n-1}(X_{t_{i+1}^n} - X_{t_i^n})^2\Big) =: \Phi_{m,n}.$$

where $Z_{m,n} = \sup_{0\le j\le k_m} \sup_{i: t_j^m \le t_i^n < t_{j+1}^m} |\xi_{n,i} - \xi_{m,j}|$

From the continuity of f'' and (5.31) it follows that $Z_{m,n} \xrightarrow{m,n\to\infty} 0$ a.s.

Moreover, as $\sum_{i=0}^{k_n-1}(X_{t_{i+1}^n} - X_{t_i^n})^2 \xrightarrow{n\to\infty} \langle X\rangle_t$ in probability,

we see that for every $\varepsilon>0$ we can choose m s.t. for all $n>m$

(5.34) $$P[\Phi_{m,n} \ge \varepsilon] \le \varepsilon.$$

Fixing this value of m, using again the approximation of $\langle X\rangle$,

$$\lim_{n\to\infty} \sum_{j=0}^{k_m-1} \xi_{m,j} \sum_{i: t_j^m \le t_i^n < t_{j+1}^m} (X_{t_{i+1}^n} - X_{t_i^n})^2 = \sum_{j=0}^{k_m-1} \xi_{m,j}(\langle X\rangle_{t_{j+1}^m} - \langle X\rangle_{t_j^m})$$

$$= \int_0^t h_m(s)\,d\langle X\rangle_s$$

where $h_m(s) = \xi_{m,j}$ if $t_j^m \le s < t_{j+1}^m$. Obviously $h_m(s) \xrightarrow{m\to\infty} f''(X_s)$ a.s.

So for m large enough

(5.35) $$P\Big[\Big|\int_0^t h_m(s)\,d\langle X\rangle_s - \int_0^t f''(s)\,d\langle X\rangle_s\Big| \ge \varepsilon\Big] \le \varepsilon.$$

Hence, combining (5.34), (5.35), (5.33), for $n>m$ large enough

$$P\Big[\Big|\sum_{i=0}^{k_n-1} \xi_{n,i}(X_{t_{i+1}^n} - X_{t_i^n})^2 - \int_0^t f''(X_s)\,d\langle X\rangle_s\Big| \ge 3\varepsilon\Big] \le 3\varepsilon$$

proving (5.32)

We now treat the general case $K>1$. Taylor formula yields

$$f(X^1_{t^n_{i+1}},\dots,X^K_{t^n_{i+1}}) - f(X^1_{t^n_i},\dots,X^K_{t^n_i}) = \sum_{k=1}^{K} \frac{\partial f}{\partial x_k}(X^1_{t^n_i},\dots,X^K_{t^n_i})(X^k_{t^n_{i+1}} - X^k_{t^n_i})$$

$$+ \sum_{k,l=1}^{K} \frac{\xi^{k,l}_{n,i}}{2} (X^k_{t^n_{i+1}} - X^k_{t^n_i})(X^l_{t^n_{i+1}} - X^l_{t^n_i})$$

with $\xi^{k,l}_{n,i} = \frac{\partial f}{\partial x_k \partial x_l}(s)$ for some $s \in \{(\theta X^1_{t^n_i} + (1-\theta)X^1_{t^n_{i+1}},\dots,\theta X^K_{t^n_i} + (1-\theta)X^K_{t^n_{i+1}}) : \theta\in[0,1]\}$

Proposition (5.27) again gives the required result for the terms with first derivative, and similar arguments as previously show that for all k,l

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} \xi^{k,l}_{n,i} (X^k_{t^n_{i+1}} - X^k_{t^n_i})(X^l_{t^n_{i+1}} - X^l_{t^n_i}) = \int_0^t \frac{\partial f}{\partial x_k \partial x_l}(X^1_s,\dots,X^K_s)\, d\langle X^k, X^l\rangle_s.$$

□

Some easy consequences of Itô's formula:

- Integration by parts formula: If X,Y are semimartingales, then

(5.36) $$X_tY_t = X_0Y_0 + \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \langle X,Y\rangle_t$$

- Taking $X=Y$ in (5.36) yields

(5.37) $$X_t^2 = X_0^2 + 2\int_0^t X_s\,dX_s + \langle X\rangle_t$$

Hence, if X is local mart., we know that $X_t^2 - \langle X\rangle_t$ is also local mart. and (5.37) shows that it equals $X_0^2 + 2\int_0^t X_s\,dX_s$. This is not surprising, as our construction of $\langle X\rangle$ used the discrete approximation of the stoch. integral $\int X_s\,dX_s$.

- For a $\mathcal{G}_t$-BM (B_t), possibly started not in 0, Itô's formula gives

(5.38) $$f(B_t) = f(B_0) + \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,ds.$$

(5.39) $$f(t,B_t) = f(0,B_0) + \int_0^t \frac{\partial f}{\partial x}(s,B_s)\,dB_s + \int_0^t \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)(s,B_s)\,dt.$$

and for d-dimensional BM $B=(B^1,\dots,B^d)$

(5.40) $$f(B_t) = f(B_0) + \sum_{i=1}^{d}\int_0^t \frac{\partial f}{\partial x_i}(B_s)\,dB^i_s + \frac{1}{2}\int \Delta f(B_s)\,ds$$

(as $\langle B^i, B^j\rangle_t = \delta_{ij}t$).

(5.38) <u>Remark</u>: (5.37) can be actually used to prove Itô's formula.
See Rogers & Williams or my ETH notes.

5.4. <u>Applications of Itô's formula</u>

<u>Exponential martingales</u>

(5.39) <u>Proposition</u>: Let M be a local mtg. Then for all $\lambda \in \mathbb{C}$
$$\mathcal{E}(\lambda M)_t := \exp\{\lambda M_t - \tfrac{\lambda^2}{2}\langle M\rangle_t\}$$
is a local mtg.

<u>Proof</u>: Let $f(x,y) = \exp\{\lambda x - \frac{\lambda^2}{2} y\}$, $x,y \in \mathbb{R}$. Then

(5.40)
$$\frac{\partial f}{\partial y} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0.$$

By Itô's formula for $(X^1, X^2) = (M, \langle M\rangle)$, noting that $\langle M\rangle$ is of finite variation and thus $\langle M, \langle M\rangle\rangle = 0 = \langle\langle M\rangle\rangle$,
$$f(M_t, \langle M\rangle_t) = f(M_0, 0) + \int_0^t \partial_x f(M_s, \langle M\rangle_s)\, dM_s + \int_0^t \partial_y f(M_s, \langle M\rangle_s)\, d\langle M\rangle_s$$
$$+ \frac{1}{2}\int_0^t \partial_x^2 f(M_s, \langle M\rangle_s)\, d\langle M\rangle_s$$
$$\overset{(5.40)}{=} f(M_0, 0) + \int_0^t \partial_x f(M_s, \langle M\rangle_s)\, dM_s \quad .$$
Since $\mathcal{E}(\lambda M) = f(M, \langle M\rangle)$, this shows that $\mathcal{E}(\lambda M)$ is a local mtg. $\square$.

It is important to know, e.g. for applications of Girsanov's theorem, under which conditions $\mathcal{E}(M)$ is a true martingale.

(5.41) <u>Theorem</u> (Novikov's condition) Let M be a local mtg with $M_0 = 0$. Consider the following properties

(i) $E[\exp\{\frac{1}{2}\langle M\rangle_\infty\}] < \infty$

(ii) M is UI and $E[\exp\{\frac{1}{2} M_\infty\}] < \infty$

(iii) $\mathcal{E}(M)$ is UI

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

<u>Proof</u>: <u>(i) $\Rightarrow$ (ii)</u>: By (i), we have also $E[\langle M\rangle_\infty] < \infty$, so, by (4.51), M is a true mtg bounded in L^2, so M is UI. Moreover,
$$\exp \tfrac{1}{2} M_\infty = \mathcal{E}(M)_\infty^{1/2} \exp\{\tfrac{1}{2}\langle M\rangle_\infty\}^{1/2}$$
so by Cauchy-Schwarz inequality

$$E\left[\exp\left\{\tfrac{1}{2}M_\infty\right\}\right] \le \underbrace{E\left[\mathcal{E}(M)_\infty\right]^{1/2}}_{\overset{(4.39)}{\le} E[\mathcal{E}(M)_0]^{1/2}<\infty} \underbrace{E\left[\exp\left\{\tfrac{1}{2}\langle M\rangle_\infty\right\}\right]^{1/2}}_{<\infty \text{ by assumption}} < \infty$$

proving (ii).

(ii) $\Rightarrow$ (iii): As M is UI, we have $M_t = E[M_\infty | \mathcal{G}_t]$, by (4.21). Hence, $\exp \frac{1}{2} M_t \le E[\exp \frac{1}{2} M_\infty \mid \mathcal{G}_t]$, by Jensen inequality. As consequence $\exp\frac{1}{2}M_t \in L^1$. Applying the Jensen's inequality again, we see that $\exp\frac{1}{2}M_t$ is a submartingale closed by its limit $\exp\frac{1}{2}M_\infty$. Using the stopping theorem for submartingales, we have

$$\exp\tfrac{1}{2}M_T \le E[\exp\tfrac{1}{2}M_\infty \mid \mathcal{G}_T] \quad \text{for all st. times } T$$

(5.42) i.e. the family $\{\exp\frac{1}{2}M_T : T \text{ is st. time}\}$ is UI.

Let now for $a \in (0,1)$, $Z_t^{(a)} = \exp\left\{\frac{a M_t}{(1+a)}\right\}$. Then

$$\mathcal{E}(aM)_t = \exp\left\{aM_t - \tfrac{a^2}{2}\langle M\rangle_t\right\} \quad \text{and as} \quad a = a^2 + \frac{a(1-a^2)}{1+a}$$

$$= \mathcal{E}(M)_t^{a^2}\left(Z_t^{(a)}\right)^{1-a^2}.$$

For $\Gamma \in \mathcal{G}_\infty$, T stopping time we have thus by Hölder inequality

$$E[1\!\!1_\Gamma \mathcal{E}(aM)_T] = E[\mathcal{E}(M)_T]^{a^2} E[1\!\!1_\Gamma Z_T^{(a)}]^{1-a^2} \quad (\mathcal{E}(M) \text{ is supermartg.})$$

$$\le E[1\!\!1_\Gamma Z_T^{(a)}]^{1-a^2} = E[1\!\!1_\Gamma Z_T^{(a)}]^{2a(1-a)\underbrace{\tfrac{(1+a)}{2a}}_{\ge 1}}$$

$$\overset{\text{Jensen}}{\le} E[1\!\!1_\Gamma \exp\tfrac{1}{2}M_T]^{2a(1-a)}$$

(5.42) then implies that $\{\mathcal{E}(aM)_T : T \text{ is st. time}\}$ is UI, in particular $\mathcal{E}(aM)$ is a UI martg. Hence, by Hölder again

$$1 = E[\mathcal{E}(aM)_\infty] \le E[\mathcal{E}(M)_\infty]^{a^2} E[Z_\infty^{(a)}]^{1-a^2}$$

$$\le E[\mathcal{E}(M)_\infty]^{a^2} E[\exp\tfrac{1}{2}M_\infty]^{2a(1-a)}$$

As $a \to 1$, this implies $E[\mathcal{E}(M)_\infty] \ge 1$ and so $E[\mathcal{E}(M)_\infty] = 1$, since $\mathcal{E}(M)$ is supermartg. But this implies that $\mathcal{E}(M)$ is UI martg, by (4.21). $\square$

Lévy characterisation of BM.

Recall that an $\mathbb{R}^d$ valued process X with $X_0 = 0$ is called $\mathcal{G}_t$-BM if for all $0 \le s < t$, $X_t - X_s$ is independent of $\mathcal{G}_s$ and $\mathcal{N}(0,(t-s)I)$.

(5.43) **Theorem** (Lévy) Let $X = (X^1, \dots, X^d)$ be $\mathcal{G}_t$-adapted. Then TFAE

(i) X is d-dimensional $\mathcal{G}_t$-BM

(ii) $X^1, \dots, X^d$ are continuous $\mathcal{G}_t$-local mgs, $X_0^i = 0$, and

$$\langle X^i, X^j \rangle_t = \delta_{ij}\, t \qquad \forall 1 \le i,j \le d \;\; \forall t \ge 0.$$

Proof: (i) ⇒ (ii) is already known

(ii) ⇒ (i): Let $\xi \in \mathbb{R}^d$. Then $\langle \xi \cdot X \rangle_t = \sum_{i,j} \xi_i \xi_j \langle X^i, X^j \rangle_t = |\xi|^2 t$. ← scalar product in $\mathbb{R}^d$

By (5.39) we thus see that

$$Z_t = \exp\left\{ i \xi \cdot X_t + \tfrac{1}{2} |\xi|^2 t \right\}$$

is a local mg which is bounded when t remains bounded, i.e. it is a true martingale. Hence for $0 \le s \le t$, $E[Z_t | \mathcal{G}_s] \overset{a.s.}{=} Z_s$.

Since $|Z_t| > 0$, we can rewrite this as

$$1 \overset{a.s}{=} E\left[\frac{Z_t}{Z_s} \,\Big|\, \mathcal{G}_s \right] = E\left[\exp\left\{ i\xi \cdot (X_t - X_s) + \frac{|\xi|^2}{2}(t-s) \right\} \Big| \mathcal{G}_s \right]$$

and thus

$$E\left[\exp\{ i \xi \cdot (X_t - X_s)\} \,|\, \mathcal{G}_s \right] = \exp\left\{ -\frac{|\xi|^2}{2}(t-s) \right\}.$$

This implies that $(X_t - X_s)$ is indep. of $\mathcal{G}_s$ and $\mathcal{N}(0,(t-s)I)$, that is X is $\mathcal{G}_t$-BM □.

As an application of Lévy's theorem we now show that, "modulo time change", the BM is the most general cont. local mtg.

(5.44) <u>Theorem</u> (Dubins - Schwarz 1965)

Let M be a local mrtg started from 0 such that $\lim_{t\to\infty} \langle M\rangle_t = \infty$.
For $t \geq 0$ define the stopping times

$$\tau_t = \inf\{s : \langle M\rangle_s > t\} \nearrow \infty \quad \text{as } t\to\infty,$$

set $\mathcal{H}_t = \mathcal{G}_{\tau_t} = \{A \in \mathcal{G} : A \cap \{\tau_t \leq s\} \in \mathcal{G}_s\}$. Then
$B_t := M_{\tau_t}$ is a $\mathcal{H}_t$-BM. Moreover, for every fixed t, $\langle M\rangle_t$ is a $\mathcal{H}_t$-st. time and

$$M_t = B(\langle M\rangle_t), \quad t \geq 0.$$

<u>Remark:</u> The hypothesis $\langle M\rangle_t \nearrow \infty$ is unessential. If it does not hold, B_t is defined only up to time $\langle M\rangle_\infty$ (which is random)

<u>Proof:</u> M is continuous and thus progressively measurable. Hence, by (3.49), $B_t = M_{\tau_t}$ is $\mathcal{H}_t = \mathcal{G}_{\tau_t}$-measurable, i.e. B is $\mathcal{H}_t$-adapted.
Observe also that $\mathcal{H}_t$ satisfies the usual conditions. ($\to$ Exercise)
The function $t \mapsto \tau_t$ is cadlag non-decreasing and

$$\lim_{s\to t-} \tau_s = \tau_{t-} = \inf\{s \geq 0 : \langle M\rangle_s = t\}.$$

To show the continuity of B we need

(5.45) <u>Lemma:</u> The intervals of constancy of M coincide a.s. with the intervals of constancy of $\langle M\rangle$, that is P-a.s. for $\forall a<b$

$$M_t = M_a \ \forall t\in[a,b] \iff \langle M\rangle_t = \langle M\rangle_a \ \forall t \in [a,b].$$

<u>Proof:</u> $\Rightarrow$ follows easily by approximation of $\langle M\rangle$, as usual.
$\Leftarrow$ Consider local mrtg $(0,b-a)\ni t\mapsto M_{(a+t)} - M_a$. The quadratic variation of this process is 0 by assumption, so it is constant by (4.54). $\square$

Now observe that for $t\in[\tau_{s-}, \tau_s]$ we have $\langle M\rangle_t = s$, so P-a.s for $s \geq 0$

$$\lim_{r\to s-} B_r = \lim M_{\tau_r} = M_{\tau_{s-}} = M_{\tau_s} = B_s,$$

so B is left-continuous. Their right-continuity then follows from this of τ.

We now show that B_t and $B_t^2 - t$ are $\mathcal{H}_t$ - local mgs.
For every $n \geq 1$, the stopped local mgs M^{τ_n}, $(M^{\tau_n})^2 - \langle M \rangle^{\tau_n}$ are then UI martingales (using (4.51) and $\langle M^{\tau_n} \rangle_\infty = \langle M \rangle_{\tau_n} = n$).
The stopping theorem then yields for $r \leq s \leq n$

$$E[B_s \mid \mathcal{H}_r] = E[M^{\tau_n}_{\tau_s} \mid \mathcal{G}_{\tau_r}] = M^{\tau_n}_{\tau_r} = B_r$$

$$E[B_s^2 - s \mid \mathcal{H}_r] = E[(M^{\tau_n}_{\tau_s})^2 - \langle M^{\tau_n} \rangle_{\tau_s} \mid \mathcal{G}_{\tau_r}] = (M^{\tau_n}_{\tau_r})^2 - \langle M^{\tau_n} \rangle_{\tau_r} = B_r^2 - r.$$

Using Lévy's theorem (5.43) then implies that B is $\mathcal{H}$-BM.
To see that $\langle M \rangle_t$ is $\mathcal{H}$-stopping time, we write $\{\langle M \rangle_t \leq u\} = \{\tau_u \geq t\} \in \mathcal{H}_u$
The last claim follows by inverting the definition of B, using (5.45) □

Bernstein's inequality:

Another application of the exponential mgs yields the following inequality which also confirms that $\langle M \rangle$ is "the right clock" for M again.

(5.46) Theorem: Let M_t be a cont. local mg with $M_0 = 0$ and $\langle M \rangle_t \leq ct$ for all $t \geq 0$ and some $c < \infty$. Then M is a mg and

$$P\Big[\sup_{s \leq t} M_s \geq a\Big] \leq \exp\Big\{-\frac{a^2}{2ct}\Big\} \quad \text{for all } a > 0,\ T > 0.$$

(and thus $P\big[\sup_{s \leq t} |M_s| \geq a\big] \leq 2\exp\big\{-\frac{a^2}{2ct}\big\}$)

Proof. For $\lambda \in \mathbb{R}$, due to (5.39), (5.41), $Z_t = e^{\lambda M_t - \frac{\lambda^2}{2}\langle M \rangle_t}$ is a true martingale. For $\lambda > 0$, by Doob's inequality, using $\langle M \rangle_t \leq ct$,

$$P\Big[\sup_{s \leq t} M_s \geq a\Big] \leq P\Big[\sup_{s \leq t} Z_s \geq e^{\lambda a - \frac{\lambda^2}{2} ct}\Big]$$
$$\leq e^{-\lambda a + \lambda^2 ct/2} \underbrace{E[Z_t]}_{= E[Z_0] = 1}.$$

Setting $\lambda = \frac{a}{ct}$ implies the first inequality. The second follows by taking $-M$ instead of M. The fact that M is a mg follows from (5.41) or from the second inequality.

Burkholder - Davis - Gundy inequalities

The following important inequalities have a similar flavour.

(5.47) **Theorem**: For every $p>0$ there are constants $c_p, C_p \in (0,\infty)$ s.t. for every local mg M with $M_0=0$
$$c_p \, E[\langle M\rangle_\infty^{p/2}] \le E[(\sup_{s\ge 0}|M_s|)^p] \le C_p \, E[\langle M\rangle_\infty^{p/2}].$$

(5.47) **Remark**: By considering, for T a st. time, stopped local mg M^T, we obtain an analogous inequality
$$c_p \, E[\langle M\rangle_T^{p/2}] \le E[(\sup_{s\le T}|M_s|)^p] \le C_p \, E[\langle M\rangle_T^{p/2}].$$

Proof: We only consider $p\ge 2$. For $p\in(0,2)$ see Revuz-Yor sec. IV.4.

- upper bound: By Itô's formula applied on $f(x)=|x|^p$ we obtain

(5.48) $$|M_t|^p = \int_0^t p|M_s|^{p-1}\,\mathrm{sign}\, M_s \, dM_s + \frac{1}{2}p(p-1)\int_0^t |M_s|^{p-2}\, d\langle M\rangle_s.$$

(Observe that for $p\ge 2$, $f\in C^2(\mathbb{R},\mathbb{R})$)

Assume first that $M, \langle M\rangle$ are bounded. Then $M\in H^2$, $p|M_s|^{p-1}\,\mathrm{sign}\, M_s \in L^2(M)$ and so the first term in (5.48) is a true martingale in H^2. Hence, Doob's inequality yields
$$\begin{aligned}
E[(\sup_{s\le t}|M_s|)^p] &\le \left(\frac{p}{p-1}\right)^p E[|M_t|^p] \\
&\overset{(5.48)}{\le} \mathrm{const}(p)\, E\left[\int_0^t |M_s|^{p-2}\, d\langle M\rangle_s\right] \\
&\le \mathrm{const}(p)\, E\left[(\sup_{s\le t}|M_s|)^{p-2}\langle M\rangle_t\right] \\
&\overset{\text{Hölder}}{\le} \mathrm{const}(p)\, E\left[(\sup_{s\le t}|M_s|)^p\right]^{1-\frac{2}{p}} E\left[\langle M\rangle_t^{\frac{p}{2}}\right]^{\frac{2}{p}}
\end{aligned}$$
where we applied Hölder with "$\frac{1}{r}=1-\frac{2}{p}$" and "$\frac{1}{s}=\frac{2}{p}$".

Rearranging and letting $t\to\infty$ gives the upper bound when $M, \langle M\rangle$ are bounded.

For a general local mg M, we proceed by localisation. Set $T_n = \inf\{s\ge 0: |M_s|\ge n \text{ or } \langle M\rangle_s \ge n\}$. Then the upper bound holds for M^{T_n} and thus for M, by MCT on both sides.

- <u>Power bound:</u> By localisation we again assume that $M, \langle M \rangle$ are bounded. Consider

$$Y_t = \delta + \varepsilon \langle M \rangle_t + M_t^2 = \delta + (1+\varepsilon)\langle M_t \rangle + 2\int_0^t M_s \, dM_s, \quad \delta, \varepsilon > 0, t \geq 0.$$

By Itô's formula on $f(x) = x^m$, i.e. $f \in C^2[[\delta, \infty), \mathbb{R}]$, for $m = \frac{p}{2}$

$$Y_t^m = \delta^m + m(1+\varepsilon)\int_0^t Y_s^{m-1} d\langle M_s \rangle + 2m \int_0^t M_s Y_s^{m-1} dM_s$$
$$+ 2m(m-1)\int_0^t Y_{m-2} M_s^2 \, d\langle M_s \rangle$$

Since $M, \langle M \rangle$ are bounded and $Y \geq \delta$, $2m\int_0^t M_s Y_s^{m-1} dM_s$ is a bounded mart., so its expectation vanishes. Hence

$$E Y_t^m = \delta^m + m(1+\varepsilon) E \int_0^t Y_s^{m-1} d\langle M \rangle_s + 2m(m-1) E \int_0^t Y_s^{m-2} M_s^2 \, d\langle M \rangle_s$$

The last term is non-negative. Taking now $\delta \downarrow 0$, we obtain

$$E\left[(\varepsilon \langle M \rangle_t + M_t^2)^m\right] \geq m(1+\varepsilon) E\left[(\varepsilon \langle M \rangle_s + M_s^2)^{m-1} d\langle M \rangle_s\right]$$
$$\geq m(1+\varepsilon)\varepsilon^{m-1} E \int_0^t \langle M \rangle_s^{m-1} d\langle M \rangle_s$$
$$= (1+\varepsilon)\varepsilon^{m-1} E\left[\langle M \rangle_t^m\right].$$

For $m \geq 1$, x^m is convex on $[0, \infty)$, so $2^{m-1}(x^m + y^m) \geq (x+y)^m$.
Hence, the last inequality can be written as

$$\varepsilon^m E \langle M \rangle_t^m + E\left[|M_t|^{2m}\right] \geq (1+\varepsilon)\left(\frac{\varepsilon}{2}\right)^{m-1} E \langle M \rangle_t^m.$$

and thus

$$E\left[|M_t|^{2m}\right] \geq \underbrace{\left[(1+\varepsilon)\left(\frac{\varepsilon}{2}\right)^{m-1} - \varepsilon^m\right]}_{c(m) > 0 \text{ if } \varepsilon \text{ is taken small}} E\left[\langle M \rangle_t^m\right]$$

Since $E\left[|M_t|^{2m}\right] \leq E\left[(\sup_{s \leq t} |M_s|)^{2m}\right]$, this completes the proof, after taking $t \to \infty$. $\square$.

5.5 Brownian motion & harmonic functions

In this short section we derive two important properties of d-dimensional BM ($d \geq 2$). As we will see they can be proved by application of Itô's formula (see (5.40)).

(5.49) **Definition:** Let $U \in \mathbb{R}^d$ be a non-empty open set. A function $f \in C^2(U;\mathbb{R})$ is called harmonic (in U) if $\Delta f(x) = \left(\sum_1^d \partial_i^2 f(x)\right) = 0$ for all $x \in U$.

Harmonic function plays important role in the study of BM.

(5.50) **Proposition:** When $d \geq 2$ and $x \neq 0$ is a point of $\mathbb{R}^d$, then W_x-a.s. $\{X_t \neq 0$ for all $t \geq 0\}$.
(Here we work in the setting of canonical BM).

Remark: Last proposition implies that "BM does not hit points when $d \geq 2$".

Proof: For $g \in C^2((0,\infty);\mathbb{R})$ we define the radial function

$$f(x) = g(|x|) = g\left(\sqrt{x_1^2 + \dots + x_d^2}\right), \qquad x \in \mathbb{R}^d \setminus \{0\}.$$

The following identity can be checked easily:

(5.51) $$\Delta f(x) = g''(r) + \frac{d-1}{r} g'(r) \quad \text{with } r = |x|, \text{ for } x \in \mathbb{R}^d \setminus \{0\}.$$

For $d \geq 3$ we choose $g(r) = r^{2-d}$ so that

$$g''(r) + \frac{d-1}{r} g'(r) = (2-d)(1-d)\, r^{-d} + (d-1)(2-d)\, r^{-d} = 0.$$

and thus

(5.52) $$f(x) = \frac{1}{|x|^{d-2}}, \quad x \neq 0, \text{ is harmonic on } \mathbb{R}^d \setminus \{0\}.$$

For $x \neq 0$ we fix $0 < a < |x| < b < \infty$ and pick a smooth function f_a which satisfies $f_a = f$ on $\{x \in \mathbb{R}^d : |x| \geq a\}$. We apply Itô's formula (5.29): For W_x-a.s. for $t \geq 0$

$$f_a(X_t) = f_a(X_0) + \int_0^t \nabla f_a(X_s)\, dX_s + \frac{1}{2} \int_0^t \Delta f_a(X_s)\, ds.$$

Introducing the stopping time

$$\tau = \inf\{s \geq 0 : |X_s| \geq b \text{ or } |X_s| \leq a\}$$

we see that W_x-a.s.

$$|X_{t\wedge\tau}|^{2-d} = f_a(X_{t\wedge\tau}) = |x|^{2-d} + \int_0^{t\wedge\tau} \nabla f_a(X_s)\,dX_s + \frac{1}{2}\int_0^{t\wedge\tau} \Delta f_a(X_s)\,ds$$

As the last term vanishes since $\Delta f_a(X_s)=0$ for $s\le t\wedge\tau$, we see that $(|X_{t\wedge\tau}|^{2-d})_{t\ge 0}$ is a bdd. local martingale and thus $(|X_{t\wedge\tau}|^{2-d})_{t\ge 0}$ is a martingale.

Therefore,

$$E[\,|X_{t\wedge\tau}|^{2-d}] = |x|^{2-d} \qquad \text{for all } t\ge 0.$$

Letting $t\to\infty$, using DCT, we find

$$|x|^{2-d} = E[\,|X_\tau|^{2-d}] = a^{2-d}\,W_x(|X_\tau|=a) + b^{2-d}\,W_x(|X_\tau|=b).$$

Using then $W_x(|X_\tau|=a) + W_x(|X_\tau|=b) = 1$ we obtain

(5.53) $$W_x(|X_\tau|=a) = \frac{|x|^{2-d}-b^{2-d}}{a^{2-d}-b^{2-d}}.$$

Letting $a\to 0$ while keeping b we see

$$W_x(H_{\{0\}} < H_{B(0,b)^c}) = 0,$$

where $H_A = \inf\{s\ge 0 : X_s\in A\}$. Then it follows that

$$W_x(X_t=0 \text{ for some } t) = W_x(H_{\{0\}}<\infty) = \lim_{b\to\infty} W_x(H_{\{0\}} < H_{B(0,b)^c}) = 0.$$

and Proposition is proved for $d\ge 3$.

For $d=2$ we choose instead $g(r) = \log\frac{1}{r}$. Again

$g''(r) + \frac{d-1}{r}g'(r) = 0$ and

(5.54) $f(x) = \log\frac{1}{|x|}$ is harmonic on $\mathbb{R}^2\setminus\{0\}$.

The same reasoning as above then yields

(5.55) $$W_x(|X_\tau|=a) = \frac{\log\frac{b}{|x|}}{\log\frac{b}{a}}$$

which then yield the Proposition by sending $a\to 0$ and then $b\to\infty$. □

Similar ideas can be applied for discussing the recurrence and transience of BM.

(5.56) <u>Theorem:</u> (transience of BM in $d\ge 3$)

When $d\ge 3$ then for $x\in\mathbb{R}^d$, W_x-a.s. $\lim_{t\to\infty}|X_t| = \infty$.

<u>Proof:</u> As for $x,z\in\mathbb{R}^d$, under W_x the process $(X_t+z)_{t\ge 0}$ is a BM started from $x+z$, it suffices to show (5.56) for some $x\ne 0$.

As previously we see that (for $0<a<|x|$)

$|X_{t\wedge H(B(0,a))}|^{2-d}$ is a cont. bounded martingale under W_x.

Using Fatou's lemma for conditional expectations, for $s\le t$, W_x-a.s.

$$E_x\left[|X_t|^{2-d}\mid \mathcal{F}_s\right] = E_x\left[\liminf_{n\to\infty} |X_{t\wedge H_{B(0,\frac{1}{n})}}|^{2-d}\mid \mathcal{F}_s\right]$$
$$\overset{\text{Fatou}}{\le} \liminf_{n\to\infty} E_x\left[|X_{t\wedge H_{B(0,\frac{1}{n})}}|^{2-d}\mid \mathcal{F}_s\right]$$
$$= \liminf_{n\to\infty} |X_{s\wedge H_{B(0,\frac{1}{n})}}|^{2-d} \overset{(5.2)}{=} |X_s|^{2-d}.$$

Hence, $(|X_t|^{2-d})_{t\ge 0}$ is a cont. supermartg under W_x.
Since this supermartingale is non-negative, it follows from the supermartg. convergence theorem that

(5.57) W_x-a.s. $|X_t|^{2-d}$ has a finite limit as $t\to\infty$.

On the other hand, by looking on one component of X, we already know that

(5.58) W_x-a.s. $\limsup_{t\to\infty} |X_t| = \infty$.

(5.57) and (5.58) then together imply (5.56) □.

We now return to the two-dimensional BM.

(5.59) <u>Theorem:</u> (recurrence of BM in $\mathbb{R}^2$). When $d=2$, for any $x\in\mathbb{R}^d$ W_x-a.s., for any non-empty open set $O\subset\mathbb{R}^2$, the set $\{t\ge 0,\ X_t\in O\}$ is unbounded.

<u>Proof:</u> From (5.55) we see by letting $b\to\infty$ that when $a<|x|$, then W_x-a.s. $H_{B(0,a)}<\infty$. This of course remains true when $|x|\le a$, so

(5.60) for any $x\in\mathbb{R}^2$, $a>0$, W_x-a.s. $H_{B(0,a)}<\infty$.

We can then define a sequence of stopping times (w.r.t. $\mathcal{F}_t$):

$S_1 = H_{B(0,a)}$, $S_2 = S_1\circ\theta_{S_1+1} + S_1 + 1$ and inductively

$S_{i+1} = S_1\circ\theta_{S_i+1} + S_i + 1$.

It follows that $S_i \nearrow \infty$. Using the strong Markov property, for $y\in\mathbb{R}^2$

(5.61) $$W_y[S_{i+1}<\infty] = W_y\left[S_i<\infty,\ \theta^{-1}_{S_i+1}(S_1<\infty)\right]$$
$$= E_y\Big[S_i<\infty,\ \underbrace{P_{X_{S_i+1}}[S_1<\infty]}_{=1 \text{ by } (5.60)}\Big] = W_y[S_i<\infty].$$
$$\overset{\text{induction}}{=} W_y[S_1<\infty] = 1.$$

Hence, by construction for $i \geq 1$, W_y-a.s. on $\{S_i < \infty\}$, $X_{S_i} \in \overline{B}(0,a)$ and thus using (5.61) and $S_i \nearrow \infty$, for any $y \in \mathbb{R}^2$ W_y-a.s. for any a, the set $\{t \geq 0 : X_t \in \overline{B}(0,a)\}$ is unbounded.

Using that W_y is the law of $(X_t + y)$ under W_0 we get W_0-a.s. for all $z \in \mathbb{Q}^2$, $n \geq 1$, $\{t \geq 0, X_t \in \overline{B}(z, \frac{1}{n})\}$ is unbounded, which proves the theorem for $x = 0$. The general case follows by the translation again. □.

(5.62) <u>Remark:</u> The last theorem implies that the trajectory of BM (run for an infinite amount of time), $\{X_t : t \geq 0\}$, is dense in $\mathbb{R}^2$.

(5.63) <u>Exercise:</u> Use martingale convergence theorem and the recurrence of BM in $d=2$ to show Liouville's theorem, that is that every bounded harmonic function on $\mathbb{R}^2$ is constant.

<u>Conformal invariance of BM:</u>

Itô's formula can be used to show the following striking property of Brownian motion path in $\mathbb{R}^2$. In this section we use the identification $\mathbb{R}^2$ and $\mathbb{C}$ and use complex notation when convenient.

As motivation consider the following examples.

(a) Let X be canonical BM in $d=2$. For $\theta \in \mathbb{R}$ consider the rotation $f_\theta(x) = e^{i\theta} x$ of $\mathbb{C}$, and extend it to $C = C([0,\infty); \mathbb{C})$ by $f_\theta(w)_t = f_\theta(w_t)$, $w \in C$. Then the image measure of W_0 by f_θ is again W_0, $W_0 \circ f_\theta^{-1} = W_0$. ($\longrightarrow$ exercise)

(b) Consider now map $\phi(z) = az + b$ for $a \neq 0$, $b \in \mathbb{C}$. Using the scaling property of BM, we obtain that, under W_0,

$$\phi(X_t) = B_{|a|^2 t}$$

where B_t is a BM on $\mathbb{R}^2$ distributed according to W_b.

In particular image of the Brownian path under ϕ is again a (time-change) of Brownian path. We will now see that this is true even if $\phi(z) \sim az + b$ only locally.

Recall that a function $\varphi: \mathbb{C} \to \mathbb{C}$ is called holomorphic in $U \subset \mathbb{C}$, for U an open subset of $\mathbb{C}$, if the complex derivative $\lim_{z \to z_0} \frac{\varphi(z) - \varphi(z_0)}{z_0 - z}$ exists for every $z_0 \in U$. Every holomorphic function is analytic and, due to Cauchy-Riemann equations both real and imaginary part of φ are harmonic.

(5.64) <u>Theorem</u>: Let $U \subset \mathbb{C}$ open, $z_0 \in U$, $\varphi: U \to \mathbb{C}$ holomorphic and let B be a Brownian motion started from z_0. Set

$$T_U = H_{U^c} = \inf\{s \geq 0 : B_s \notin U\} \leq +\infty.$$

Then there exists a complex BM $\overline{B}$ such that for $t = [0, \tau_u)$

$$\varphi(B_t) = \overline{B}_{C_t}$$

where

$$C_t = \int_0^t |\varphi'(B_s)|^2 \, ds.$$

<u>Proof</u>: Write $\varphi = g + ih$, so that g, h are harmonic on U.
By Itô's formula applied to $g(B_t^1 + iB_t^2)$ we get for $t < T_U$

$$g(B_t) = g(z_0) + \int_0^t \frac{\partial g}{\partial x}(B_s)\, dB_s^1 + \int_0^t \frac{\partial g}{\partial y}(B_s)\, dB_s^2$$

and similarly

$$h(B_t) = h(z_0) + \int_0^t \frac{\partial h}{\partial x}(B_s)\, dB_s^1 + \int_0^t \frac{\partial h}{\partial y}(B_s)\, dB_s^2.$$

This shows that $M_t = g(B_t)$, $N_t = h(B_s)$ are cont. local martingales on the stoch. interval $[0, T_U)$.

Cauchy-Riemann equations $\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}$, $\frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}$ imply then

$$\langle M \rangle_t = \int_0^t \left(\frac{\partial g}{\partial x}(B_s)\right)^2 + \left(\frac{\partial g}{\partial y}(B_s)\right)^2 ds = \int_0^t |\varphi'(B_s)|^2 \, ds = C_t.$$

and similarly, using the properties of the stoch. integral,

$$\langle N \rangle_t = C_t \quad \text{and} \quad \langle M, N \rangle_t = 0.$$

Setting $\tau_t = \inf\{u : C_t > t\}$, it follows from Dubins-Schwarz theorem that $\overline{B}_t^1 = M_{\tau_t}$, $\overline{B}_t^2 = N_{\tau_t}$ are Brownian motions, Moreover, $\langle M, N \rangle_t = 0$ and a slight extension of the argument of the first half of 15/3 implies that $\langle \overline{B}^1, \overline{B}^2 \rangle = 0$, which by Lévy's theorem yields that $\overline{B} = (\overline{B}^1, \overline{B}^2)$ is 2D Brownian motion, and $\varphi(B_t) = \overline{B}_{C_t}$ on $[0, T_U)$.

Remark that $\overline{B}^1, \overline{B}^2$ are defined only on stoch. interval $[0, C_{T_U})$, but they can be "arbitrarily" extended to all $\mathbb{R}_+$. $\square$

(5.68) Remark: (c.f. (4.51), (4.52)) The previous theory provides us with an example of a local martingale M which is bounded in L^2, i.e. $\sup_{t\geq 0} \|M_t\|_{L^2(P)} < \infty$, but is not a true martingale.

Important counterexample!

To this end fix $d=3$ and consider a d-dim BM B_t started from $x \neq 0$ and set $M_t = \frac{1}{|B_t|}$.

By arguments of the proof of (5.50), we know that M_t is a local martingale. Moreover, as

$$P[B_t \in dy] = \frac{1}{(2\pi t)^{3/2}} e^{-\frac{(y-x)^2}{2t}} =: p_t(x,y)$$

We have for $t\geq 0$, for a $\varepsilon < |x|$ small

$$E[M_t^2] \leq \frac{1}{\varepsilon^2} P[|B_t| > \varepsilon] + \int_{|y|\leq\varepsilon} p_t(x,y)\, y^{-2}\, dy$$

$$\leq \frac{1}{\varepsilon^2} + \underbrace{\sup_{|y|\leq\varepsilon} p_t(x,y)}_{<\infty \text{ unif in } t} \underbrace{\int_{|y|\leq\varepsilon} y^{-2} dy}_{<\infty \text{ as we are in } \mathbb{R}^3} \leq C < \infty \text{ for all } t\geq 0.$$

Hence M is L^2 bounded and thus UI. If M would be martingale, then $M_t = E[M_\infty \mid \mathcal{F}_t]$, but we already proved that $M_\infty = 0$ a.s., leading to contradiction.

Observe that M is UI local mart which is not mart.

Exercise: Prove that $E[\langle M\rangle_\infty] = \infty$, cf (4.51(ii))

5.7. Change of measure & Girsanov transformation

In this section we study how do semimartingales on a probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$, satisfying the usual conditions, behave under a different probability measure Q on the same space. Q will be chosen absolutely continuous wrt P.

(5.66) Lemma: Let $Q \ll P$ on $\mathcal{G}_\infty := \sigma(\bigcup_{t\geq 0} \mathcal{G}_t)$ $(\neq \mathcal{G}$ in general$)$ and let

$$D_t = \frac{dQ}{dP}\Big|_{\mathcal{G}_t}, \quad t \in [0,\infty]$$

be the Radon-Nikodym derivative of Q wrt P on $\mathcal{G}_t$. Then D is a UI $\mathcal{G}_t$-martingale, which we can assume being cadlag. With this assumption, also $D_T = \frac{dQ}{dP}\big|_{\mathcal{G}_T}$ for every st. time T. If Q is equivalent to P on $\mathcal{G}_\infty$ (i.e. $Q(A)=0 \Leftrightarrow P(A)=0 \ \forall A \in \mathcal{G}_\infty$) then also

$$P[\,\forall t\geq 0 \ D_t > 0 \text{ and } D_{t-} > 0\,] = 1.$$

Proof. For $A \in \mathcal{G}_t$ we have

$$Q(A) = E^Q[1\!\!1_A] = E^P[1\!\!1_A D_\infty] = E^P[1\!\!1_A E^P[D_\infty \mid \mathcal{G}_t]]$$

As the Radon-Nikodym derivative is ess. unique, it follows that $D_t = \frac{dP}{dQ}\big|_{\mathcal{G}_t} = E[D_\infty \mid \mathcal{G}_t]$. Hence D is UI martingale closed by D_∞. The cadlag replacement is possible due to (4.19).

Further, for a stopping time T, we have by (4.22) for $A \in \mathcal{G}_T$

$$Q(A) = E^P[1\!\!1_A D_\infty] = E^P[1\!\!1_A D_T]$$

As D_T is $\mathcal{G}_T$ measurable, this implies that $D_T = \frac{dQ}{dP}\big|_{\mathcal{G}_T}$.

To show the last claim, let $T = \inf\{t : D_t = 0 \text{ or } D_{t-} = 0\}$

We claim that

(5.67) $$P[\,D_t = 0 \ \forall t \geq T\,] = 1$$

Indeed, set $T_\varepsilon = \inf\{t : D_t \leq \varepsilon\}$. Then $T_\varepsilon \nearrow T$ as $\varepsilon \downarrow 0$.

Hence, for $t \in \mathbb{Q}$, by stopping theorem

$$P[D_{t\vee T_\varepsilon} > \sqrt{\varepsilon}] \le \frac{1}{\sqrt{\varepsilon}} E[D_{t\vee T_\varepsilon}] = \frac{1}{\sqrt{\varepsilon}} E[D_{T_\varepsilon}] \le \sqrt{\varepsilon}.$$

Letting $\varepsilon \downarrow 0$ implies that $P[D_{t\vee T} = 0] = 1$ and thus $P[D_t = 0 \;\forall t \in \mathbb{Q}, t \ge T] = 1$ which yields (5.67) by cadlag property. As Q and P are equivalent, we know that $P[D_t = 0] = 0$. Comparing this with (5.67) then yields $P[T = \infty] = 1$, completing the proof □.

The next lemma links D from (5.66) to exp. martingales of (5.39)

(5.68) <u>Lemma:</u> Let D be a <u>continuous</u> strictly positive local mtg. Then there is a unique continuous local mtg L such that

$$D_t = \exp\{L_t - \tfrac{1}{2}\langle L\rangle_t\} = \mathcal{E}(L)$$

L is given by

$$L_t = \log D_0 + \int_0^t D_s^{-1}\, dD_s$$

Moreover,

$$D_t = D_0 + \int_0^t D_s\, dL_s$$

<u>Proof:</u> Uniqueness follows by standard arguments: if L and L' satisfy the defining relation, then $\langle L - L'\rangle_t = 0$.

To see the second property, we use $D_t > 0$ and apply Ito on $\log D_t$

$$\log D_t = \log D_0 + \int_0^t \frac{dD_s}{D_s} - \frac{1}{2}\int_0^t \frac{d\langle D\rangle_s}{D_s^2} = L_t - \frac{1}{2}\langle L\rangle_t$$

The last property holds for every pair $L, D = \mathcal{E}(L)$. It follows directly from the proof of (5.39).

(5.69) <u>Theorem</u> (Cameron-Martin 1944, Girsanov 1960)
Let Q be equivalent with P on $\mathcal{G}_\infty$. Assume that the process D of (5.66) is continuous. Let L be as in (5.68).
Then, if M is $(\mathcal{G}_t, P)$-local mart., then its <u>Girsanov transform</u>

$$\tilde{M}_t = M_t - \langle M, L\rangle_t, \quad t \geq 0$$

is a $(\mathcal{G}_t, Q)$-local mart.

<u>Proof:</u> We first show

(5.70) <u>Claim:</u> If T is a st. time and X a cont. adapted process such that $(XD)^T$ is a P-martingale, then X^T is a Q-martingale.

<u>Proof.</u> Observe

$$E^Q[|X_{t\wedge T}|] \overset{(5.66)}{=} E^P[|X_{t\wedge T} D_{t\wedge T}|] < \infty, \text{ since } (XD)^T \text{ is } P\text{-mart.}$$

That is $X_t^T \in L^1(Q)$ for all $t \geq 0$. Take $A \in \mathcal{G}_s$, $s < t$.
Then $A \cap \{T > s\} \in \mathcal{G}_s$, so, as $(XD)^T$ is P-mart.

$$E^P[1\!\!1_{A\cap\{T>s\}} X_{T\wedge t} D_{T\wedge t}] = E^P[1\!\!1_{A\cap\{T>s\}} X_{T\wedge s} D_{T\wedge s}]$$

Since $D_{T\wedge s} = \frac{dQ}{dP}\big|_{\mathcal{G}_{T\wedge s}}$, $D_{T\wedge t} = \frac{dQ}{dP}\big|_{\mathcal{G}_{T\wedge t}}$. This implies

$$E^Q[1\!\!1_{A\cap\{T>s\}} X_{T\wedge t}] = E^Q[1\!\!1_{A\cap\{T>s\}} X_{T\wedge s}]$$

On the other hand, trivially

$$E^Q[1\!\!1_{A\cap\{T\leq s\}} X_{T\wedge t}] = E^Q[1\!\!1_{A\cap\{T\leq s\}} X_{T\wedge s}]$$

so $E^Q[1\!\!1_A X_t^T] = E^Q[1\!\!1_A X_s^T]$, proving the claim $\square$

As consequence of (5.70) we see easily that if XD is a cont. P-local mart., then X is a continuous Q-local mart.
Taking now M a P-local mart., using $X = \tilde{M}$, Ito formula yields

$$\tilde{M}_t D_t = M_0 D_0 + \int_0^t \tilde{M}_s \, dD_s + \int_0^t D_s \, dM_s - \int_0^t D_s \, d\langle M, L\rangle_s + \langle M, D\rangle_t$$

$$= M_0 D_0 + \int_0^t \tilde{M}_s \, dD_s + \int_0^t D_s \, dM_s$$

since by (5.68) $d\langle M, L\rangle_s = D_s^{-1} d\langle D, M\rangle_s$. I.e. $\tilde{M}D$ is P-local mart. and then $\tilde{M}$ is Q-local mart.

(5.71) <u>Corollary</u>: When Q is equivalent with P on $\mathcal{G}_\infty$, then the families of P-semimartingales and Q-semimartingales coincide.

<u>Proof</u>: Let Y be a P-semimart. with decomposition $Y_t = Y_0 + M_t + A_t$. Then, by (5.69),
$$Y_t = Y_0 + \tilde{M}_t + (A_t + \langle M, L\rangle_t)$$
with Q-local mart. $\tilde{M}_t$. Hence, Y is Q-semimartingale.
As the assumptions in the theorem are symmetric in Q, P □

(5.72) <u>Remark</u>: To go back from Q-local mart., to P-local mart. we observe that $\frac{dP}{dQ}\Big|_{\mathcal{G}_t} = D_t^{-1}$, and $D_t^{-1} = \mathcal{E}(-L)$
So if M_t is Q local mart., then by (5.69), $\tilde{M}_t = M_t + \langle M, L\rangle$ is P-local mart.

(5.73) <u>Corollary</u>: Let P, Q be equivalent on $\mathcal{G}_\infty$ and X, Y two P-semimartingales (and thus Q-semimartingales). Then P-a.s.
$$\langle X, Y\rangle_t^P = \langle X, Y\rangle_t^Q \qquad \forall t \geq 0$$
and $\langle X, Y\rangle.$ is given by the approximation (4.65).

<u>Proof</u>: It suffices to use (4.65) and observe that if P and Q are equivalent then the limits in P- and Q-probability must coincide □.

(5.74) <u>Corollary</u>: If P, Q are equivalent on $\mathcal{G}_\infty$, X semimartingale, H loc. bdd. Then the stoch. integrals $H \cdot X$ under P and Q coincide.
<u>Proof</u>: Obviously they coincide on elementary functions and as $L^2(P)$ limits and $L^2(Q)$ limits coincide too, the claim is proved □.

If the martingale M is a BM, Girsanov theorem yields:

(5.75) Theorem: Let M be a P-Brownian motion. Then $\tilde{M} = M - \langle M, L \rangle$ is Q-Brownian motion. Moreover if there is a progr. measurable process b such that

$$L_t = \int_0^t b_s \, dM_s$$

Then under Q, M is a "BM with drift"

$$M_t = \tilde{M}_t + \int_0^t b_s \, ds$$

Proof: Under Q, $\tilde{M}_t$ is a cont. local mtg with $\tilde{M}_0 = 0$ and by (5.73) $\langle \tilde{M} \rangle_t = \langle M \rangle_t = t$. Hence $\tilde{M}$ is Q-BM by Lévy's theorem (5.43). To prove the second claim, observe that

$$\tilde{M}_t = M_t - \langle M, L \rangle_t = M_t - \langle M, \int_0^{\cdot} b_s \, dM_s \rangle_t = M_t - \int_0^t b_s \, d\langle M \rangle_s$$

and M is BM, i.e. $d\langle M \rangle_t = dt$. □

(5.76) Remark: In applications of Girsanov's theorem, one usually starts with a measure P and P-local mtg L with $L_0 = 0$. One then defines $D_t = \mathcal{E}(L)_t$ which is positive local mtg, i.e. supermartingale. If D_t is a true UI martingale, one then can define Q on $\mathcal{G}_\infty$ via $Q = D_\infty \cdot P$. To ensure that this is possible, Novikov's condition is important, cf (5.41).

(5.77) Remark: The assumption on continuity of D might appear restrictive, but it is not in many situations. E.g. in the setting of canonical BM one has
Theorem: Let $(\Omega, \mathcal{G}, \mathcal{G}_t, P) = (C[0,\infty), \mathcal{F}, \mathcal{F}_t, W_0)$. Then every $\mathcal{F}_t$-local mtg M is continuous and can be written as $M_t = \int_0^t H_s \, dX_s$ for a permissible process H_s (and the canonical BM X)

5.8. Representation of martingales

We are going to show the representation theorem mentioned in Remark (5.77). We consider here the canonical space of Brownian motion, i.e. $\Omega = C([0,\infty))$, $\mathcal{F}, \mathcal{F}_t$ are as in (4.2), W is Wiener measure and X the canonical process. (i.e. BM).

(5.78) Theorem: For every random variable $Z \in L^2(C, \mathcal{F}, W)$ there is a (essentially) unique process $H \in L^2(X)$ (c.f. (5.9)) such that

$$Z = E[Z] + \int_0^\infty H_s \, dX_s.$$

The proof is based on the following lemma.

(5.79) Lemma: The vector space generated by random variables

$$\exp\{ i \sum_{j=1}^n \lambda_j (X_{t_j} - X_{t_{j-1}})\}, \quad 0 = t_0 < t_1 < \dots < t_n, \ \lambda_i \in \mathbb{R}, n \in \mathbb{N}$$

is dense in $L^2_{\mathbb{C}}(C, \mathcal{F}, W)$.

Proof: It is sufficient to show that if $Z \in L^2_{\mathbb{C}}(C, \mathcal{F}, W)$ satisfies

(5.80) $$E[Z \exp(i \sum_{j=1}^n \lambda_j (X_{t_j} - X_{t_{j-1}}))] = 0 \qquad \forall t_i, \lambda_i, n,$$

then $Z = 0$ W-a.s., which is equivalent to the fact that the measure $(ZW)(d\omega) := Z(\omega) W(d\omega)$ on $(C, \mathcal{F})$ is a zero measure.

To this end we first show that it is a zero measure on $(C, \sigma(X_{t_0}, \dots, X_{t_n}))$ for a fixed sequence $0 = t_0 < \dots < t_n$. (5.80) says that the image of ZW by the map $\omega \mapsto (X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}})$ is the zero measure, since its Fourier transform is zero. I.e. this measure vanishes on $\sigma(X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}) = \sigma(X_{t_0}, \dots, X_{t_n})$.

We now know that $W[Z 1_A] = 0$ for every $A \in \sigma(X_{t_0}, \dots, X_{t_n})$. A Dynkin's type argument then implies the lemma. $\square$

Proof of Theorem (5.78)

Let $\mathcal{H}$ be the subspace of $Z \in L^2(C, \bar{\mathcal{F}}, W)$ which can be written as stated. For $Z \in \mathcal{H}$ we have

(5.81) $$E[Z^2] = E[Z]^2 + E\left[\int_0^\infty H_s^2 \, ds\right]$$

In particular, this implies the uniqueness of the representation. (if H' would be another process representing Z, then $E[\int_0^\infty (H_s^2 - (H_s')^2) ds] = 0$)

On the other hand, if $Z_n \in \mathcal{H}$ is a Cauchy sequence in $L^2(C, \bar{\mathcal{F}})$, then the associated processes H_n form a Cauchy sequence in $L^2(X)$, by (5.81). As $L^2(X)$ is complete, $H_n \to H \in L^2(X)$ and, by properties of the stoch. integral, $Z_n \xrightarrow{L^2(W)} Z = \int_0^\infty H \, dX_s \in \mathcal{H}$, i.e. $\mathcal{H}$ is complete.

Finally, taking $0 = t_0 < t_1 < \ldots < t_n$, $\lambda_i \in \mathbb{R}$, $n \in \mathbb{N}$, defining
$$f(s) = \sum_j \lambda_j \mathbb{1}_{(t_{j-1}, t_j]}(s)$$
and using $\mathcal{E}_t^f$ to denote the exp. martingale $\mathcal{E}(i \int_0^\cdot f(s) dX_s)$, we obtain by Ito's formula that
$$\exp\left(i \sum \lambda_j (X_{t_j} - X_{t_{j-1}}) + \frac{1}{2} \sum \lambda_j^2 (t_j - t_{j-1})\right) = \mathcal{E}_\infty^f = 1 + \int_0^\infty f(s) \mathcal{E}_s^f \, dX_s .$$
Hence, the r.v.'s of the form $\operatorname{Re}/\operatorname{Im}\left(\exp\left(i \sum \lambda_j (X_{t_j} - X_{t_{j-1}})\right)\right)$ are in $\mathcal{H}$. By (5.79) such r.v. are dense in $L^2(C, \bar{\mathcal{F}}, W)$, that is $\mathcal{H} = L^2(C, \bar{\mathcal{F}}, W)$ $\square$.

(5.82) Theorem: Every $(\mathcal{F}_t)$- local mtg M has a version written as
$$M_t = C + \int_0^t H_s \, dX_s$$
for a constant C and $H \in L^2_{loc}(X)$ (If M is L^2 bdd, then $H \in L^2(X)$).

In particular, every $(\mathcal{F}_t)$-local mtg has a continuous version.

Proof: If M is bounded in L^2, then $M_\infty \in L^2(C, \bar{\mathcal{F}})$, so by (5.78)
$$M_\infty = EM_\infty + \int_0^\infty H_s \, dX_s \quad \text{with} \quad H \in L^2(X).$$
Also

$$M_t = E[M_\infty \mid \bar{\mathcal{F}}_t] = E[M_\infty] + E\left[\int_0^\infty H_s\,dX_s \mid \bar{\mathcal{F}}_t\right] =$$
$$= EM_\infty + \int_0^t H_s\,dX_s$$

so the theorem holds true in this case.

If M is a local mtg, then first $M_0 = C \in \mathbb{R}$ by Blumenthal 0-1 law. Taking $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$, using the first part of the proof, there is $H_n \in L^2(X)$ s.t.

$$M_t^{T_n} = C + \int_0^t H_n(s)\,dX_s.$$

Using the uniqueness of the representation, for $m<n$,
$H_n(s,\omega) = H_m(s,\omega)$ ds-a.e. W-a.s. on $[0,T_m]$
We can thus construct $H \in L^2_{loc}(X)$ such that
$H(s,\omega) = H_m(s,\omega)$ ds-a.e. W-a.s. on $[0,T_m]$
By construction of the stoch. integral we then have $M_t = C + \int_0^t H_s\,dX_s$ as required. $\square$

(5.83) <u>Remark:</u> As $\langle M,X\rangle_t = \int_0^t H_s\,ds$, H is Radon-Nikodym derivative of $d\langle M,X\rangle_s$ w.r.t. the Lebesgue measure. In most concrete examples it is hard to write H explicitly.

5.9. <u>Wiener chaos expansion (*)</u>

We give another representation theorem for $L^2(C,\mathcal{F},W)$.
Let

$$\Delta_n = \{(s_1,\dots,s_n) \in \mathbb{R}^n : s_1 > s_2 \dots > s_n\}$$

and $L^2(\Delta_n) := L^2(\Delta_n, dx)$. Define

$$E_n := \left\{ f : \Delta_n \to \mathbb{R} : f = \prod_{i=1}^n f_i(s_i),\ f_i \in L^2(\mathbb{R}) \right\}$$

The set of linear combinations of elements of E_n is dense in $L^2(\Delta_n)$. ($\to$ Exercise) For $f \in E_n$, set

$$J_n(f) = \int_0^\infty f_1(s_1)\,dX_{s_1} \underbrace{\int_0^{s_1} f_2(s_2)\,dX_{s_2} \dots \int_0^{s_{n-1}} f_n(s_n)\,dX_{s_n}}_{:= F_{n-1}(s_1)}$$

Observe that

$$\|J_n(f)\|_2^2 = E\left[\int_0^\infty f_1^2(s_1)\, F_{n-1}^2(s_1)\, ds_1\right] = \int_0^\infty f_1^2(s_1)\, E F_{n-1}^2(s_1)\, ds_1$$

$$\overset{\text{inductively}}{=} \int_0^\infty f_1^2(s_1) ds_1 \int_0^{s_1} f_2^2(s_2) ds_2 \cdots \int_0^{s_{n-1}} f_n^2(s_n)\, ds_n = \|f\|^2_{L^2(\Delta_n)}$$

(5.54) <u>Definition</u>: The smallest closed linear subspace K_n of $L^2(C, \bar{\mathcal{F}}, W)$ containing $J_n(E_n)$ is called the n-th Wiener Chaos.

By the previous observation, J_n can be extended to an isometry of $L^2(\Delta_n)$ and K_n which is one-to-one. Moreover, if $f \in E_n$, $g \in E_m$ for $m < n$, then

$$E^W[J_n(f) J_m(g)] = E\left[\int_0^\infty f_1(s_1) g_1(s_1)\, F_{n-1}(s_1) G_{m-1}(s_1)\, ds_1\right]$$

$$= \int_0^\infty f_1(s_1) g_1(s_1) ds_1 \int_0^{s_1} f_2(s_2) g_2(s_2)\, ds_2 \cdots \int_0^{s_{m-1}} f_m(s_m) g_m(s_m) ds_m$$

$$\times\, E\left[\int_0^{s_m} f_{m+1}(s_{m+1})\, dX_{m} \cdots \int_0^{s_{n-1}} f_n(s_n)\, dX_n\right]$$

$$= 0.$$

so K_m, K_n are orthogonal in $L^2(C, \bar{\mathcal{F}}, W)$.

(5.55) <u>Theorem</u>: $L^2(C, \bar{\mathcal{F}}, W) = \bigoplus_{n=0}^\infty K_n$, where K_0 is the space of constants. Otherwise said, $Z \in L^2(C, \bar{\mathcal{F}}, W)$ can be written as

$$Z \overset{L^2}{=} E[Z] + \sum_{i=1}^\infty J_n(f_n) \qquad f_n \in L^2(\Delta_n).$$

<u>Proof</u>.

We start by a technical lemma. For its statement, let h_n be n-th Hermite polynomial given by

$$\sum_{n\geq 0} \frac{u^n}{n!} h_n(x) = \exp\left\{ux - \frac{u^2}{2}\right\} \qquad u, x \in \mathbb{R}$$

i.e.

$$h_n(x) = e^{\frac{x^2}{2}} (-1)^n \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$$

Set $\quad H_n(x,a) = a^{n/2} h_n(x/\sqrt{a}), \; H_n(x,0) = x^n$. Then

$$\exp\left(ux - \frac{au^2}{2}\right) = \sum_{n\geq 0} \frac{u^n}{n!} H_n(x, a)$$

(5.86) <u>Lemma</u>: Let M be a local mtg and $M_0 = 0$. Then

$$L_t^{(n)} = H_n(M_t, \langle M\rangle_t) \text{ is a local mtg.}$$

and $$L_t^{(n)} = n! \int_0^t dM_{s_1} \int_0^{s_1} dM_{s_2} \dots \int_0^{s_{n-1}} dM_{s_n}$$

<u>Proof</u>: One sees easily that $\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial a} H_n(x,a) = 0$.
and $\frac{\partial H_n}{\partial x} = n H_{n-1}$. Hence by Itô's formula

$$L_t^{(n)} = n \int_0^t L_s^{(n-1)} dM_s$$

This proves the representation of $L_t^{(n)}$ and thus the claim □.

Consider now the martingale $M_t = \sum_{j=1}^n \lambda_j (X_{t_j \wedge t} - X_{t_{j-1}\wedge t})$
with $\langle M\rangle_t = \sum_{j=1}^n \lambda_j^2 (t_j \wedge t - t_{j-1} \wedge t)$. Then

(5.87) $$\exp\{i \sum \lambda_j (X_{t_j} - X_{t_{j-1}}) + \frac{1}{2}\sum_{j=1}^n \lambda_j^2 (t_j - t_{j-1})\} = \sum_{n\geq 0} \frac{(i)^n}{n!} H_n(M_\infty, \langle M\rangle_\infty)$$

and every $H_n(M_\infty, \langle M_\infty\rangle)$ can be written as in Lemma (5.86).

with $\lambda_{n+1} = 0$, $t_{n+1} = \infty$

Moreover, $dM_s = \prod_{j=1}^{n+1} (1 + (\lambda_j - 1) 1_{(t_{j-1}, t_j]}(s)) dX_s =: f(s) dX_s$,
that is

$$L_t^{(n)} = n! \int_0^t f(s_1) dX_1 \int_0^{s_1} f(s_2) dX_2 \dots \int_0^{s_{n-1}} f(s_n) dX_n,$$

proving that the RHS of (5.87) can be written as

(5.88) $$1 + \sum_{j=1}^\infty J_n(f_n) \quad \text{with} \quad f_n(s_1, \dots, s_n) = i^n \prod_{i=1}^n f(s_i)$$

The equality in (5.88) holds pointwise, and since f is bdd with compact support also in L^2.

Hence, the statement of the theorem is true for r.v. as on the RHS of (5.87). Using Lemma (5.79) the claim follows □.

Chapter VI: STOCHASTIC DIFFERENTIAL EQUATIONS

The goal of stochastic differential equations (SDE's) is to provide a mathematical model for evolution of a physical system subjected to an external noise.

Let us start with an ODE

(6.1) $$x'(t) = b(x(t))$$

which describes the development of a system with a local drift b. Under suitable assumptions on b it is known that the solution of this ODE exists and is unique (loc. or globally).

We now want to construct a model that locally around a point x behaves like a solution to (6.1) subjected to a Brownian noise, i.e. it looks locally like

(6.2) $$x + b(x)\,t + \sigma(x)\,B(t)$$

We will directly consider the vector case, that is $x \in \mathbb{R}^d$, $b(x) \in \mathbb{R}^d$, $\sigma(x)$ a $(d \times n)$-matrix and B_t a BM in $\mathbb{R}^n$ introducing the noise.

In other words, the increments of our process behave infinitesimally like a Gaussian r.v. with mean $b(x)dt$ and covariance matrix $a(x)$ given by

(6.3) $$\xi^T a(x)\,\xi = E\left[\left(\xi^T \cdot \sigma(x)\,(B_{t+dt} - B_t)\right)^2\right] = E\left[\left((\sigma(x)^T\xi)^T \cdot (B_{t+dt} - B_t)\right)^2\right]$$
$$= |\sigma(x)^T \xi|^2\,dt = \xi^T \sigma(x)\,\sigma(x)^T \xi, \quad \text{for } \xi \in \mathbb{R}^d$$

Hence $\quad a(x) = \sigma(x)\,\sigma(x)^T \qquad (d\times d\text{-matrix}).$

One way to construct such model is to consider a SDE

(6.4) $$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t\,, \quad X_0 = x_0$$

The meaning of (6.4) is given by its integral form

(6.5) $$X_t = x_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s$$

or, coordinate wise

(6.6) $$X_t^i = x_0^i + \int_0^t b_i(X_s)\,ds + \sum_{j=1}^n \int_0^t \sigma_{ij}(X_s)\,dB_s^j\,, \quad i = 1, \dots, d.$$

Observe that (6.5), (6.6) are well defined, given the construction of the stochastic integral, and that X is a semimartingale.

In many situations it is useful that b, σ depend on t explicitly, i.e. to consider the following generalisation of (6.5)

(6.7) $$X_t = x_0 + \int^t b(s, X_s)\,ds + \int^t \sigma(s, X_s)\,dB_s.$$

We now define the solutions of this SDE.

(6.8) <u>Definition</u> Let $d, m \in \mathbb{N}$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to M_{d\times m}(\mathbb{R})$, $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be measurable, loc. bdd. A solution to the equation

$E(\sigma, b)$ $$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$$

is a collection:

- a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ sat. the usual conditions
- a m-dimensional $\mathcal{G}_t$ - Brownian motion B
- a $\mathcal{G}_t$-adapted continuous process $X = (X^1, \dots, X^d)$ such that

(6.9) $$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s$$

If $X_0 = x_0 \in \mathbb{R}^d$, we say that X solves $E_{x_0}(\sigma, b)$.

There are various notions of the existence and the uniqueness of the solutions to $E(\sigma, b)$.

(6.10) <u>Definition:</u> We say that for the SDE $E(\sigma, b)$ there is

(a) <u>weak existence</u> if for every $x \in \mathbb{R}^d$ there is a solution to $E_x(\sigma, b)$

(b) <u>weak existence & uniqueness</u>: if for every $x \in \mathbb{R}^d$ there is solution X to $E_x(\sigma, b)$ and all such solutions have the same law.

(c) <u>pathwise uniqueness</u> if given $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ and B any two solutions X, X' s.t. $X_0 = X_0' = x_0$ are indistinguishable

(d) a solution to $E(\sigma, b)$ is said <u>strong</u> if it is adapted to the (completed) filtration of the BM B.

Examples: Suppose we have a probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ satisfying the usual conditions and β_t a $\mathcal{G}_t$-BM, $d = n = 1$

(6.11) • weak existence: $dX_t = 2X_t^{1/2} dB_t + dt$, $X_0 = 0$

is solved by $B_t = \beta_t$, $X_t = B_t^2$

which can be checked easily by Itô's formula.

(6.12) • weak uniqueness: Let $dX_t = f(X_t)\, dB_t$ for $f: \mathbb{R} \to \mathbb{R}$ measurable with $|f(x)| = 1$ $\forall x \in \mathbb{R}$.

Then $X_t = \int_0^t f(X_s)\, dB_s$ and thus X_t is a martingale with $\langle X \rangle_t = \int_0^t \underbrace{f(X_s)^2}_{=1} ds = t$. By Lévy's theorem, X is a BM

(6.13) • Point-wise uniqueness: the following equation describes so called geometric BM (with drift)

$$dX_t = \sigma X_t\, dB_t + \mu X_t\, dt, \quad X_0 = 1.$$

It is solved by $B = \beta$ and $X_t = \exp\{\sigma B_t + (\mu - \frac{\sigma^2}{2})t\}$, as can easily be proved by Itô's formula (cf. exponential mtgs)

For sake of simplicity, consider $\sigma = 1$, $\mu = 0$. Let X, Y be two solutions on a given probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ with a BM B, i.e. $dX_t = X_t\, dB_t$, $dY_t = Y_t\, dB_t$, $X_0 = Y_0 = 1$. Then, by Itô

$$\frac{d}{dt}\left(\frac{Y_t}{X_t}\right) = \frac{dY_t}{X_t} - \frac{Y_t\, dX_t}{X_t^2} + \frac{Y_t}{X_t^3} d\langle X \rangle_t - \frac{1}{X_t^2} d\langle X, Y \rangle_t$$

$$= \frac{Y_t\, dB_t}{X_t} - \frac{Y_t X_t\, dB_t}{X_t^2} + \frac{Y_t}{X_t^3} X_t^2\, dt - \frac{1}{X_t^2} X_t Y_t\, dt = 0.$$

It follows easily that X and Y are indistinguishable.

(6.14) • weak existence, no weak uniqueness: Consider

$$dX_t = 3\,\mathrm{sign}(X_t)\,|X_t|^{2/3} dB_t + 3\,\mathrm{sign}(X_t)\,|X_t|^{1/3} dt, \quad X_0 = 0.$$

By Itô, $\beta = B$ and $X_t = B_t^3$ solve this equation, but $X \equiv 0$ solves it as well.

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(6.15) • <u>weak existence and uniqueness, no pathwise uniqueness</u>

Let $\operatorname{sign}(x) = 1\!\!1\{x \geq 0\} - 1\!\!1\{x < 0\}$ and consider Tanaka's equation

$$dX_t = \operatorname{sign}(X_t)\, dB_t \quad , \quad X_0 = 0.$$

With $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ and β as usual, set

$$B_t = \int_0^t \operatorname{sign}(\beta_s)\, d\beta_s .$$

By Lévy's theorem, B is a BM. Then $X_t = \pm \beta_t$ solve the equation.

Indeed, $dX_t = d\beta_t = \underbrace{\operatorname{sign}(\beta_t)\operatorname{sign}(\beta_t)}_{=1} d\beta_t = \operatorname{sign} \beta_t\, dB_t = \operatorname{sign} X_t\, dB_t$

and for $X_t = -\beta_t$

$$dX_t = -d\beta_t = \underbrace{\{\operatorname{sign}(\beta_t)\operatorname{sign}(-\beta_t) - 2\, 1\!\!1_{\beta_t = 0}\}}_{=-1} d\beta_t$$

$$\overset{?}{=} \operatorname{sign}(-\beta_t)\, dB_t = \operatorname{sign} X_t\, dB_t$$

where in the equality "$\overset{?}{=}$" we ignored the second term since $\int_0^t 1\!\!1_{\beta_s = 0}\, ds = 0$ and thus also $\int_0^t 1\!\!1_{\beta_s = 0}\, d\beta_s = 0$.

We will later show that Tanaka's equation has no strong solution.

From the following theorem it follows that there are no SDE's with pathwise uniqueness and no weak uniqueness, or pathwise uniqueness and no strong solution.

(6.16) <u>Theorem:</u> (Yamada–Watanabe) If the equation $E_x(\sigma, b)$ has weak existence and pathwise uniqueness, then it also has weak uniqueness. Moreover, for every $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ and every $\mathcal{G}_t$-BM B, there is a strong solution to $E_x(\sigma, b)$ $\forall x$.

<u>Proof</u>: See Revuz–Yor, Section IX.1.

(10.14) **Theorem: (Picard's iteration method)**
Assume that $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to M_{d\times n}$ are continuous and satisfy the Lipshitz condition

(10.15) $$|b(t,y)-b(t,z)| + |\sigma(t,y)-\sigma(t,z)| \le k|y-z| \quad \forall t\in\mathbb{R}_+,\ y,z\in\mathbb{R}^d$$

and uniform growth condition

(10.16) $$|b(t,y)| + |\sigma(t,y)| \le K(1+|y|) \quad \forall t\in\mathbb{R}_+,\ y\in\mathbb{R}^d.$$

Then for any $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge0}, P)$ and $(B_t)_{t\ge0}$ as above and any $x_0\in\mathbb{R}^d$ there is a essentially unique strong solution to (10.8).

(10.17) **Remark** (a) (10.15), (10.16) are equivalent to (10,15) + uniform boundedness of $|b(t,0)|, |\sigma(t,0)|$.

(b) The continuity of b, σ in t seems not necessary for the proof below, measurability should be sufficient ($\to$ check as an exercise).

Proof of Theorem (10.14)

- uniqueness: Let X, Y be two strong solutions to (10.8). For $M > |x_0|$ define

(10.18) $$T = \inf\{u\ge 0,\ |X_u|\ge M \text{ or } |Y_u|\ge M\}.$$

Then P-a.s. for all $t\ge 0$.

(10.19) $$X_{t\wedge T} - Y_{t\wedge T} = \int_0^t (b(t,X_u)-b(t,Y_u))\,du + \int_0^t (\sigma(t,X_u)-\sigma(t,Y_u))\,dB_u.$$

Therefore, for any $t_0\ge 0$, using the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$

(10.20) $$E\Big[\sup_{t\le t_0} |X_{t\wedge T}-Y_{t\wedge T}|^2\Big] \le 2E\Big[\sup_{t\le t_0}\Big|\int_0^{T\wedge t}(b(u,X_u)-b(u,Y_u))\,du\Big|^2\Big] + 2E\Big[\sup_{t\le t_0}\Big|\int_0^{T\wedge t}(\sigma(u,X_u)-\sigma(u,Y_u))\,dB_u\Big|^2\Big].$$

By Cauchy-Schwarz inequality, the first term on the RHS of (10.20)

(10.21) $$\le 2t_0\, E\Big[\int_0^{t_0\wedge T} |b(u,X_u)-b(u,Y_u)|^2\,du\Big]$$

and using the Lipschitz condition and Fubini's theorem

(10.22) $$\le 2t_0 k^2 \int_0^{t_0} E\big[|X_{u\wedge T}-Y_{u\wedge T}|^2\big]\,du.$$

For the second term on the RHS of (10.20), by Doob's inequality with $p=2$, compound rule, we obtain the bound

$$\le 8\,E\Big[\,\Big|\int_0^{t\wedge T}(\sigma(u,X_u)-\sigma(u,Y_u))\,dB_u\Big|^2\Big]$$

(10.23) $$\overset{(6.96\,iii)}{=} 8\,E\Big[\int_0^{t\wedge T}(\sigma(u,X_u)-\sigma(u,Y_u))^2\,du\Big]$$

$$\overset{(10.15)}{\le} 8K^2\,E\Big[\int_0^{t\wedge T}|X_u-Y_u|^2\,du\Big].$$

Combining (10.20) - (10.23) we thus have

(10.24) $$E\Big[\sup_{t\le t_0}|X_{t\wedge T}-Y_{t\wedge T}|^2\Big]\le(8K^2+2t_0K^2)\int_0^{t_0}E\big[|X_{u\wedge T}-Y_{u\wedge T}|^2\big]\,du.$$

To control (10.24) we need the following useful lemma.

(10.25) <u>Lemma (Gronwall)</u> Let f be a nonnegative integrable function on $[0,t]$ such that for some $0\le a,b<\infty$ and all $0\le u\le t$

(10.26) $$f(u)\le a+b\int_0^u f(s)\,ds.$$

Then, for all $0\le u\le t$

(10.27) $$f(u)\le a\cdot\exp\{bu\}.$$

<u>Proof:</u> Iterating the inequality (10.26) we get

$$f(u)\le a+b\int_0^u f(s)\,ds$$
$$\le a+bau+b^2\int_0^u ds_1\int_0^{s_1}ds_2\,f(s_2)\le\dots$$
$$\le a+bau+\tfrac{1}{2}ab^2u^2+\dots+\tfrac{1}{n!}ab^nu^n+b^{n+1}\int_0^u ds_1\int_0^{s_1}ds_2\dots\int_0^{s_n}ds_{n+1}\,f(s_{n+1})$$
$$\le ae^{bu}+b^{n+1}\int_0^u\frac{(u-s)^n}{n!}f(s)\,ds$$
$$\le ae^{bu}+b^{n+1}\frac{u^n}{n!}\int_0^u f(s)\,ds\xrightarrow{n\to\infty}ae^{bu}.\qquad\square$$

Applying the lemma on (10.24) with $a=0$, $b=8k^2+2t_0k^2$ and $f(u)=E\big[\sup_{s\le u}|X_{s\wedge T}-Y_{s\wedge T}|^2\big]\ \big(\ge E\big[|X_{u\wedge T}-Y_{u\wedge T}|^2\big]\big)$, we obtain that

(10.28) $$E\Big[\sup_{t\le t_0}|X_{t\wedge T}-Y_{t\wedge T}|^2\Big]=0.$$

The uniqueness follows by letting $M\to\infty$ (i.e. $T\to\infty$) since then (10.28) can be read (recall t_0 is arbitrary).

$$X_t=Y_t\quad P\text{-a.s.}\quad\text{for all } t\ge 0.$$

– existence: We iteratively define for $m \geq 0$, $t \geq 0$,

$$X_t^0 = x_0$$

$$X_t^1 = x_0 + \int_0^t b(s, X_s^0)\,ds + \int_0^t \sigma(s, X_s^0)\,dB_s$$

(10.29)
$$\dots$$

$$X_t^{m+1} = x_0 + \int_0^t b(s, X_s^m)\,ds + \int_0^t \sigma(s, X_s^m)\,dB_s.$$

$$\dots$$

Then, for $m \geq 1, t \geq 0$

(10.30) $$X_t^{m+1} - X_t^m = \int_0^t \left(b(s, X_s^m) - b(s, X_s^{m-1})\right) ds + \int \left(\sigma(s, X_s^m) - \sigma(s, X_s^{m-1})\right) dB_s.$$

With $M > |x|$ and $T_M = \inf\{u \geq 0 : |X_u^m| \geq M \text{ or } |X_u^{m-1}| \geq M\}$ similarly as above we have for $0 \leq t_0 \leq t$

(10.31) $$E\Big[\sup_{s \leq t_0 \wedge T_M} |X_s^{m+1} - X_s^m|^2\Big] \leq (8 + 2t) k^2 \int_0^{t_0} E\big[|X_{u \wedge T_M}^m - X_{u \wedge T_M}^{m-1}|^2\big] du$$

Obviously, $|x_0| = \sup_{s \leq t} |X_s^0| \in L^2(P)$. By (10.15), (10.16) $b(s, x_0)$ and $\sigma(s, x_0)$ are bounded in t and thus by (10.29) also $\sup_{s \leq t} |X_s^1| \in L^2(P)$. From (10.31) with $m = 1$, letting $M \to \infty$, we see that $\sup_{s \leq t} |X_s^2| \in L^2(P)$ and iterating we get

(10.32) $$\sup_{s \leq t} |X_s^m| \in L^2(P) \qquad \text{for all } m \geq 0, t \geq 0.$$

Coming back to (10.31), we can let $M \to \infty$ and find for $0 \leq t_0 \leq t$

$$E\Big[\sup_{s \leq t_0} |X_s^{m+1} - X_s^m|^2\Big] \leq k^2 (8 + 2t) \int_0^{t_0} E\big[|X_u^m - X_u^{m-1}|^2\big] du$$

(10.33) and iterating

$$\leq \{k^2(8 + 2t)\}^m \int_0^{t_0} dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{m-1}} dt_m\, E\big[|X_{t_m}^1 - X_{t_m}^0|^2\big].$$

With (10.29) we have

$$E\big[|X_{t_m}^1 - X_{t_m}^0|^2\big] \leq 2\,|b(x_0)|^2 t_m^2 + 2E\Big[\Big(\int^{t_m} \sigma(s, x_0)\,dB_s\Big)^2\Big]$$

(10.34) $$\overset{(6.96)}{\leq} 2\,|b(x_0)|\, t_m^2 + 2 \int_0^{t_m} |\sigma(s, x_0)|^2 ds$$

$$\overset{(10.15)(10.16)}{\leq} k'(x_0, t) \cdot t_m \qquad \text{for } 0 \leq t_m \leq t.$$

In (10.33) we thus obtain

(10.35) $$E\Big[\sup_{s \leq t} |X_s^{m+1} - X_s^m|^2\Big] \leq k'(x_0, t) \frac{((8 + 2t) k^2)^m}{(m+1)!}\, t^{m+1} \qquad \text{for } t \geq 0, m \geq 0$$

This implies that for $t \geq 0$

(10.36) $$\sum_{m \geq 0} E\Big[\sup_{s \leq t} |X^{m+1}_s - X^m_s|^2 \Big]^{1/2} < \infty$$

As a consequence, P-a.s. $(X^m_\cdot)$ converges uniformly on bounded time intervals to $X^\infty_\cdot$, which can be choosed adapted and continuous. Moreover,

$$E\Big[\sup_{s \leq t} |X^\infty_s - X^m_s|^2\Big]^{1/2} \overset{\text{Fatou}}{\leq} \liminf_{p \to \infty} E\Big[\sup_{s \leq t} |X^p_s - X^m_s|^2\Big]^{1/2}$$
$$\leq \sum_{k=m}^{\infty} E\Big[\sup_{s \leq t} |X^{k+1}_s - X^k_s|^2\Big]^{1/2} \xrightarrow[m \to \infty]{} 0 \quad , \quad t \geq 0.$$

Therefore, for $t \geq 0$, P-a.s.

$$\begin{array}{ccccc} X^{m+1}_t & = x_0 + & \int_0^t f(s, X^m_s)\, ds & + & \int_0^t \sigma(s, X^m_s)\, dB_s \\ {\scriptstyle m\to\infty} \downarrow {\scriptstyle L^2(P)} & & {\scriptstyle m\to\infty} \downarrow {\scriptstyle L^2(P)} & & {\scriptstyle m\to\infty} \downarrow {\scriptstyle L^2(P)} \end{array}$$

(10.37) $$X^\infty_t = x_0 + \int_0^t f(s, X^\infty_s)\, ds + \int_0^t \sigma(s, X^\infty)\, dB_s .$$

and in view of continuity of X^∞, (10.37) holds P-a.s. for $t \geq 0$

Hence $X^\infty_\cdot$ is a solution to (10.8). □

(10.38) <u>Remark:</u> From the definition of X^m (10.29) and from the fact that P-a.s. X^m converges to X^∞ uniformly on bounded intervals we see that for each $t \geq 0$,

(10.39) $$\mathcal{F}^{X^\infty_\cdot}_t = \text{" the smallest } \sigma\text{-algebra containing all null-sets of } \mathcal{G} \text{ and making } X^\infty_s \text{ measurable for all } s \leq t \text{"}$$
$$\subseteq \mathcal{F}^B_t \quad (\text{defined analogously for } B \text{ instead of } X^\infty)$$

This implies that X is even $(\mathcal{F}^B_t)_{t \geq 0}$ adapted (which is not obvious since $\mathcal{F}^B_t \subset \mathcal{G}_t$). Intuitively, X^∞_t is a function of the noise $(B_s, s \leq t)$.

<u>Weak solutions to SDE's.</u>

To see that some conditions on f, σ are necessary in order to a strong solution exists we consider a classical example, so-called Tanaka's equation. $(d = n = 1)$

(10.40) $$dX_t = \text{sign}(X_t)\, dB_t \quad , \quad Y_0 = 0 .$$

where $\text{sign}(x) = \pm 1$ depending on whether $x > 0$ or $x \leq 0$. and

Martingale problems:

Recall the heuristic discussion on the beginning of this chapter. The second approach how to construct a process that locally looks like $x + b(x)t + \sigma(x)B_t$ is via so-called martingale problem: (even if the correspondence is less obvious here).

(10.54) Definition. Solution to a martingale problem is a probability measure P_x on $(C([0,\infty), \mathbb{R}^d), \mathcal{F})$ such that

(i) $P_x[X_0 = x] = 1$

(ii) for all $f \in C_0^2(\mathbb{R}^d)$ (C^2 functions with compact support)

(10.55) $M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds,\ t \geq 0,$

is a $(\mathcal{F}_t)$-martingale under P, with

(10.56) $Lf(y) := \frac{1}{2} \sum_{i,j=1}^d a_{ij}(y) \partial_{ij}^2 f(y) + \sum_{i=1}^d b_i(y) \partial_i f(y), \qquad y \in \mathbb{R}^d$

To see a first connection between the two approaches we show:

(10.57) Proposition: Assume that $b: \mathbb{R}^d \to \mathbb{R}^d$, $\sigma: \mathbb{R}^d \to M_{d\times n}$ be measurable locally bounded functions, $x \in \mathbb{R}^d$, and on some $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ endowed with an n-dimensional $(\mathcal{G}_t)$-BM $(B_t)_{t\geq 0}$, a cont. adapted $\mathbb{R}^d$-valued process $(X_t)_{t\geq 0}$ satisfies P-a.s, for $t \geq 0$

(10.58) $X_t = x + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s$

Then for any $f \in C^2(\mathbb{R}^d, \mathbb{R})$ the process M^f defined as in (10.55), (10.56) with $a(y) = \sigma(y)\sigma(y)^T \in M_{d\times d}$, $y \in \mathbb{R}^d$, is a continuous local martingale.

Proof: By Ito's formula, P-a.s for $t \geq 0$:

(10.59) $f(X_t) = f(X_0) + \sum_{i=1}^d \int \partial_i f(X_s)\,dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij} f(X_s)\,d\langle X^i, X^j\rangle_s.$

From (10.58) we have

(10.60) $\langle X^i, X^j\rangle_t = \langle \sum_{k=1}^n \int_0^t \sigma_{ik}(X_u)\,dB_u^k, \sum_{\ell=1}^n \int_0^t \sigma_{j\ell}(X_u)\,dB_u^\ell \rangle =$

$= \sum_{k=1}^n \int_0^t \sigma_{ik}(X_u)\,\sigma_{jk}(X_u)\,du = \int_0^t a_{ij}(X_u)\,du.$

Therefore (10.59) sounds P-a.s. for $t \geq 0$

(10.61) $$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \partial_i f(X_s)\, b_i(X_s)\, ds + \sum_{i=1}^d \sum_{k=1}^m \int_0^t \partial_i f(X_s)\, \sigma_{ik}(X_s)\, dB_s^k + \frac{1}{2} \sum_{1 \leq i,j \leq d} \int_0^t a_{ij}(X_s)\, \partial^2_{ij} f(X_s)\, ds$$

$$= f(X_0) + \int_0^t Lf(X_s)\, ds + \int_0^t \underset{d\text{-vector}}{(\nabla f(X_s))^T}\, \underset{d \times m\text{-matrix}}{\sigma(X_s)}\, \underset{m\text{-vector}}{dB_s}$$

This implies the claim of the proposition. □.

We now establish a solid link between SDE's and martingale problems.

(10.62) Theorem: (a) If on some $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ endowed with a m-dimensional $(\mathcal{G}_t)$-BM, $(B_t)_{t\geq 0}$ a cont. adapted process $(Y_t)_{t \geq 0}$ satisfies P-a.s. for $t \geq 0$

(10.63) $$Y_t = x + \int_0^t b(Y_s)\, ds + \int_0^t \sigma(Y_s)\, dB_s$$

then the law P_x of $(Y_t)_{t\geq 0}$ on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ is a solution to a Mart. problem (10.54) with $a(y) = \sigma(y)\sigma(y)^T$.

(b) Conversely, if P_x is a solution to Mart. problem (10.54), then there exists an $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ endowed with an m-dimen. BM $(B_t)_{t\geq 0}$ and a cont. adapted process $(Z_t)_{t \geq 0}$ such that

(10.64) $$Z_t = x + \int_0^t b(Z_s)\, ds + \int_0^t \sigma(Z_s)\, dB_s \quad P\text{-a.s. for } t \geq 0$$

and the law of $(Z_t)_{t\geq 0}$ is P_x.

(10.65) Remark: One should not expect Z (or Y) to be strong solutions to SDE (10.4) (recall Tanaka's equation). Here "strong" is better to be understood as in Remark (10.47).

Proof: (a) When $f \in C_0^2(\mathbb{R}^d, \mathbb{R})$, then f and all its first and second derivatives are bounded. Inspecting the proof of Proposition (10.57), this implies that under P

(10.66) $f(Y_t) - f(Y_0) - \int_0^t Lf(Y_s)\, ds$ is a $(\mathcal{G}_t)$-martingale.

Denote by P_x the law of $(Y_s)_{s \geq 0}$ on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$.

We claim that (10.66) implies that P_x solves the Mart. problem (10.54).

Indeed, take $0 \le s_0 < s_1 < \dots < s_m = s < t$, $g_0, \dots, g_m \in b\mathcal{B}(\mathbb{R}^d)$. Then

$$E^{P_x}\left[\left(f(X_t) - f(X_0) - \int_0^t (Lf)(X_u)\,du\right) g_0(X_{s_0}) \cdots g_m(X_{s_m})\right] \overset{P_x = \text{law of } Y}{=}$$

$$E\left[\left(f(Y_t) - f(Y_0) - \int_0^t (Lf)(Y_u)\,du\right) g_0(Y_{s_0}) \cdots g_m(Y_{s_m})\right] \overset{(10.66)}{=} (*) \text{ see the footnote}$$

By Dynkin's Lemma we then see that under P_x, $M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds$ is an $(\mathcal{F}_t)$-martingale for any $f \in C_0^2(\mathbb{R}^d, \mathbb{R})$. As $P_x[X_0 = x] = 1$ is obvious, P_x is a solution to MP. (10.59)

(+). We will only show (+) in a special case $n = d$ and $a(x)$ is locally elliptic, (i.e. for $\emptyset \neq U \subset \mathbb{R}^d$ open bounded exists $c(U) > 0$ such that for all $\xi \in \mathbb{R}^d$ and $y \in U$

(10.67) $$\xi^T a(y)\, \xi \ge c(U)\, |\xi|^2.$$

For the general case see D. Stroock: "Lectures on stoch. analysis: diffusion theory", page 91.

Due to (10.67) and $a(\cdot) = \sigma(\cdot)\sigma(\cdot)^T$ we see that $\sigma(\cdot)$ is invertible. For $y \in U$, $\xi \in \mathbb{R}^d$ one has

(10.68) $$|\xi|^2 = \xi^T \sigma^{-1}(y)\, a(y)\, (\sigma^{-1})^T(y)\, \xi \ge c(U)\, |(\sigma^{-1})^T(y)\, \xi|^2.$$

so that (using an explicit formula for

(10.69) $\sigma^{-1}(\cdot)$ is a locally bounded measurable

On $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F}, P_x)$ we introduce for $t \ge 0$ the σ-algebra $\mathcal{H}_t$ generated by $\mathcal{F}_t$ and by negligible sets of P_x. We set $\mathcal{G}_t = \mathcal{H}_t^+$, $t \ge 0$, so that $\mathcal{G}_t$ satisfies the usual conditions (see Ch. 5). Finally, we define

(10.70) $$M_t^i = X_t^i - X_0^i - \int_0^t b_i(X_s)\,ds, \quad i = 1, \dots, d.$$

Since for $f(y) = y^i$, $Lf(y) = b_i(y)$, we see by applying (10.59) and stopping that M_t^i are continuous $(\mathcal{G}_t)$-local mgs, ($\to$ exercise). Similarly, by choosing $f(y) = y^i y^j$, so that $Lf(y) = a_{ij}(y) + y^i b_j(y) + y^j b_i(y)$ we see that

(10.71) $$X_t^i X_t^j - X_0^i X_0^j - \int_0^t \left(a_{ij}(X_s) + X_s^j b_i(X_s) + X_s^i b_j(X_s)\right) ds$$ ~~are~~ a continuous $\mathcal{G}_t$-local mtgs under P_x.

Itô's formula implies that P_x-a.s. for $t \ge 0$

$$(*) = E\left[\left(f(Y_s) - f(Y_0) - \int_0^s (Lf)(Y_u)\,du\right) g_0(Y_{s_0}) \cdots g_m(Y_{s_m})\right].$$

(10.72) $X_t^i X_t^j = X_0^i X_0^j + \int_0^t X_s^i dX_s^j + \int_0^t X_s^j dX_s^i + \langle X^i, X^j \rangle_t$
$\overset{(10.70)}{=} X_0^i X_0^j + \int_0^t (X_s^i b_j(X_s) + X_s^j b_i(X_s)) ds + \langle M^i, M^j \rangle_t$
$+$ a cont. local mart.

Comparing (10.71), (10.72) we conclude that P-a.s.

(10.73) $\langle M^i, M^j \rangle_t = \int_0^t a_{ij}(X_s) ds$ for $t \geq 0$.

($\rightarrow$ exercise: justify this really!)

We now define for $t \geq 0$

(10.74) $B_t = \int_0^t \sigma^{-1}(X_s) dM_s$ (i.e. $B_t^i = \sum_{j=1}^d \int_0^t \sigma_{ij}^{-1}(X_s) dM_s^j$).

Then $(B_t)_{t\geq 0}$ is a cont. $\mathbb{R}^d$-valued $(\mathcal{G}_t)$-local mart. and

(10.75) $\langle B^i, B^j \rangle_t = \left\langle \sum_{k=1}^d \int_0^\cdot \sigma_{ik}^{-1}(X_s) dM_s^k, \sum_{\ell=1}^d \int_0^\cdot \sigma_{j\ell}^{-1}(X_s) dM_s^\ell \right\rangle_t$
$= \sum_{k,\ell=1}^d \int_0^t \sigma_{ik}^{-1}(X_s) \sigma_{j\ell}^{-1}(X_s)\, d\langle M^k, M^\ell \rangle_s$
$\overset{(10.73)}{=} \sum_{k,\ell=1}^d \int_0^t \sigma_{ik}^{-1}(X_s)\, a_{k\ell}(X_s)\, \sigma_{j\ell}^{-1}(X_s)\, ds$
$= \int_0^t (\sigma^{-1}(X_s) a(X_s) (\sigma^{-1})^T(X_s))_{ij}\, ds = \delta_{ij}\, t.$

(10.76) By Lévy's characterisation, $(B_t)_{t\geq 0}$ is a d-dimensional $(\mathcal{G}_t)$-Brownian motion under P_x.

Finally, using (10.74) we deduce that P_x-a.s. for $t \geq 0$.

(10.77) $\int_0^t \sigma(X_s) dB_s = \int_0^t \sigma(X_s) \sigma^{-1}(X_s) dM_s = M_t$

and thus P_x-a.s. for $t \geq 0$ (using (10.70))

(10.78) $X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s$

which implies (10.64) for $Z_t := X_t$. $\square$.

As an application of Theorems (10.14), (10.62) we obtain

(10.79) Corollary: Assume that $b: \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \to M_{d\times d}$ satisfy Lipschitz condition

$$|b(y) - b(z)| + |\sigma(y) - \sigma(z)| \leq K |y - z| \quad \forall y, z \in \mathbb{R}^d, K < \infty.$$

Then for any $x \in \mathbb{R}^d$, there is a unique solution to the martingale problem (10.59) attached to $L = \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_{ij}^2 + \sum_{i=1}^d b_i \partial_i$ with $a(\cdot) = \sigma(\cdot)\sigma^T(\cdot)$.

(one says that the martingale problem attached to L is well-posed).

Proof: <u>– existence</u> By Theorem (10.14) on any probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ endowed with a $(\mathcal{G}_t)$–Brownian motion $(B_t)_{t\geq 0}$ (e.g. $(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{F}, (\mathcal{F}_t), W_0)$ will do the job) we can construct a "solution", i.e. a continuous adapted process $(Y_t)_{t\geq 0}$ such that P-a.s. for $t \geq 0$

$$Y_t = x + \int_0^t b(Y_s)\,ds + \int_0^t \sigma(Y_s)\,dB_s.$$

From Theorem (10.62 (a)) it follows that the law of $(Y_t)_{t\geq 0}$ on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ is a solution of the martingale problem attached to L and x.

<u>– uniqueness.</u>

Assume that P_x is a solution to the martingale problem (10.54) attached to L and x. By Theorem (10.62(b)) we know that we can find a $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ and $(B_t)_{t\geq 0}$ which is a n-dimensional BM and cont. adapted process $(Z_t)_{t\geq 0}$, such that P-a.s. for $t \geq 0$

(10.80) $$Z_t = x + \int_0^t b(Z_s)\,ds + \int_0^t \sigma(Z_s)\,dB_s \quad \text{and}$$

$$P_x = \text{law of } (Z_t)_{t\geq 0} \text{ on } (C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F}).$$

As the SDE possesses a unique solution, we see as in the proof of Theorem (10.14) that P-a.s. for $t \geq 0$

(10.81) $$Z_t = X_t^\infty$$

for $(X_t^\infty)_{t\geq 0}$ being the uniform limit on bounded intervals of $(X_t^n)_{t\geq 0}$

(10.82) where $$X_t^0 = x,\quad X_t^1 = x + \int_0^t b(X_s^0)\,ds + \int_0^t \sigma(X_s^0)\,dB_s, \ \dots$$

$$X_t^{n+1} = x + \int_0^t b(X_s^n)\,ds + \int_0^t \sigma(X_s^n)\,dB_s.$$

Inspecting (10.82), we see that the law of X^n (and thus of X^∞) does not change when instead of B_t we use the canonical n-dimensional BM. Combining this observation with (10.81), we see that the law of $Z_\cdot$ (i.e. P_x) is uniquely determined. □

(10.83) Remark: The last corollary has one not very satisfactory feature: We made the assumptions on σ, b involved in the SDE (10.5) and not on b, a which are involved in the martingale problem (10.54).

It is clear that it is not sufficient to assume $a(\cdot)$ Lipschitz continuous in order to find $\sigma(\cdot)$ Lipschitz continuous such that $a(\cdot) = \sigma(\cdot)\sigma(\cdot)^T$. (take $d=1$, $a(x) = |x| \Rightarrow \sigma(x) = \sqrt{|x|}$.)

On the other hand one can show that if $a: \mathbb{R}^d \to M_{d\times d}$ is

(10.84) global elliptic, i.e. for $\varepsilon > 0$ $\xi^T a(x) \xi \geq \varepsilon |\xi|^2$ $\forall x \in \mathbb{R}^d, \xi \in \mathbb{R}^d$

(10.85) Lipschitz, i.e. $|a(y) - a(z)| \leq K |y-z|$ for $\forall y, z \in \mathbb{R}^d$.

then $a^{1/2}(\cdot)$ satisfies Lipschitz condition as well.

Other condition which implies Lipschitzity of σ is:

(10.86) $\sup_{x\in\mathbb{R}^d} |a(x)| \leq C < \infty$ and

(10.87) $x \in \mathbb{R}^d \mapsto a(x) \in M_{d\times d}$ is C^2 and $\sup_{1\leq i,j\leq d} \sup_{x\in\mathbb{R}^d} |D^2_{ij} a(x)| \leq C < \infty$.

Of course, in both these cases we obtain by application of Corollary (10.79) that (10.54) is well-posed.

Chapter 11. RELATION OF SDE'S AND PDE'S.

As announced in the Introduction we are now going to see that SDE'S can be used to provide a probabilistic representation formulas for the solutions of certain second order partial differential equations (PDE'S).

We will consider the following Poisson-Dirichlet problem:

$U \subset \mathbb{R}^d$, bounded open nonempty.

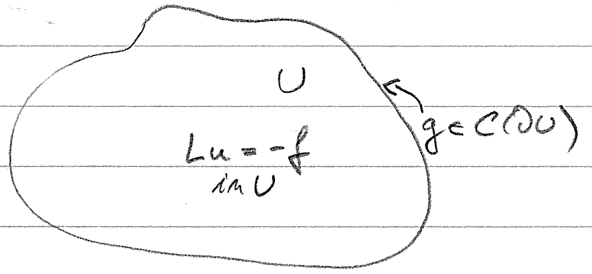
$f \in C_b(U)$, $g \in C(\partial U)$

we look for $u \in C^2(U) \cap C(\bar{U})$ such that

(11.1) $$L u(x) = -f(x) \qquad \text{for } x \in U$$
$$u(x) = g(x) \qquad \text{for } x \in \partial U$$

with as before

(11.2) $$L u(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \partial_{ij}^2 u(x) + \sum_{i=1}^{d} b_i(x) \partial_i u(x).$$
$$a(x) = \sigma(x)\sigma^T(x), \quad b : \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma : \mathbb{R}^d \to M_{d \times n}$$



The Dirichlet problem corresponds to $f \equiv 0$ and Poisson equation to $g \equiv 0$ in (11.1).

We will assume that $b(\cdot), \sigma(\cdot)$ are measurable, loc. bdd and satisfy the ellipticity condition:

(11.3) exists $c > 0$ s.t. $\xi^T a(x) \xi \geq c|\xi|$ $\forall \xi \in \mathbb{R}^d, x \in \bar{U}$.

From the theory of PDE's (c.f. Gilbarg-Trudinger, Elliptic PDE's of 2^{nd} order, page 106.) it is know that if b, σ satisfy (11.3) and the Lipschitz condition as in (10.49), f is Hölder continuous in U, and U satisfies an exterior sphere condition: $\forall z \in \partial U$ exists an open ball B with $B \cap \bar{U} = \{z\}$, then the problem (11.1) has a solution.

We now find a probabilistic representation to this solution.

(11.4) <u>Theorem:</u> (b, σ loc. bdd, meas, (11.3)) If u is a solution to (11.1) and X_t is a solution to the SDE

(11.5) $$X_t = x + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\cdot dB_s$$

for some $x \in U$, then the exit time of X from U

(11.6) $$T_U = \inf\{s \ge 0 : X_s \notin U\}$$

is P-integrable and

(11.7) $$u(x) = E\left[g(X_{T_U}) + \int_0^{T_U} f(X_s)\,ds\right].$$

<u>Proof:</u> <u>integrability of T_U:</u>

Let $\varphi(y) = C(e^{\alpha R} - e^{\alpha y_1})$ for $y = (y_1, \dots, y_d) \in \bar U$.

Then $L\varphi(y) = -Ce^{\alpha y_1}(\alpha^2 a_{11}(y) + \alpha b_1(y)) \le -Ce^{\alpha y_1}(\alpha^2 c - \alpha M)$

where c comes from (11.3) and $M := \sup_{\bar U} |b_1(y)|$.

By choosing α, R large and then C large enough we make sure

(11.8) $$L\varphi \le -1 \quad \text{on } \bar U$$

(11.9) $$\varphi > 0 \quad \text{on } \bar U.$$

Using Proposition (10.57) we know that under P

(11.10) $$\varphi(X_{t\wedge T_U}) - \varphi(x) - \int_0^{t\wedge T_U} L\varphi(X_t)\,dt \quad \text{is a local mart}$$

which is bounded and thus martingale. Therefore,

$$E[\varphi(X_{t\wedge T_U})] - \varphi(x) - E\left[\int_0^{t\wedge T_U} L\varphi(X_u)\,du\right] = 0.$$

Using now (11.8), (11.9) we find out

(11.11) $$\sup_{\bar U} \varphi \ge E[\underbrace{\varphi(X_{t\wedge T_U})}_{\ge 0}] - E\left[\int_0^{t\wedge T_U} \underbrace{L\varphi(X_u)}_{\le -1}\,du\right] \ge E[t \wedge T_U].$$

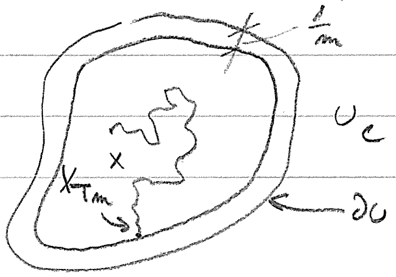
Taking $t \to \infty$ we obtain

(11.12) $$E[T_U] \le \sup_{\bar U} \varphi$$

which is enough to imply the integrability of T_U.

– <u>(11.7):</u> Fix $m \ge 1$ so that $\frac{1}{m} < d(x, U^c)$ and define

(11.13) $$T_m = \inf\{s \ge 0 : d(X_s, U_c) \le \tfrac{1}{m}\}.$$



Construct $u_m \in C_c^2(\mathbb{R}^d,\mathbb{R})$ such that

(11.14) $$u_m = u \quad \text{on} \quad \{z\in U,\ d(z,U^c)\ge \tfrac{1}{m}\}$$

By Proposition (10.57) we see that

(11.15) $$u_m(X_{t\wedge T_m}) - u_m(x) - \int_0^{t\wedge T_m} L u_m(X_s)\,ds$$
$$= u_m(X_{t\wedge T_m}) - u(x) + \int_0^{t\wedge T_m} f(X_s)\,ds$$

is a bdd. continuous martingale. Taking expectation,

(11.16) $$E[u(X_{t\wedge T_m})] - u(x) + E\left[\int_0^{t\wedge T_m} f(X_s)\,ds\right] = 0$$

As $m\nearrow\infty$ $T_m\nearrow T_U$ which is integrable we can let $t\nearrow\infty$ and then $m\nearrow\infty$ to conclude

(11.17) $$u(x) = E[u(X_{T_U})] + E\left[\int_0^{T_U} f(X_s)\,ds\right] =$$
$$= E\left[g(X_{T_U}) + \int_0^{T_U} f(X_s)\,ds\right],$$

which proves (11.7). □

Feynman-Kac formula:

A similar proof can be used to establish a relation between SDE's and parabolic PDE's. To this end we consider

(11.18) $$L_t u(x) = \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d a_{ij}(t,x)\,\partial^2_{ij} u(x) + \sum_{i=1}^d b_i(t,x)\,\partial_i u(x),$$

where $a(t,x) = \sigma(t,x)\sigma(t,x)^T$, and for a fixed $T>0$ let

(11.19) $$f(x):\mathbb{R}^d\to\mathbb{R},\quad g(t,x):[0,T]\times\mathbb{R}^d\to\mathbb{R},\quad k(t,x):[0,T]\times\mathbb{R}^d\to[0,\infty)$$

be continuous and satisfy

(11.20)
(a) $f(x)\ge 0,\ g(t,x)\ge 0 \qquad \forall x\in\mathbb{R}^d,\ 0\le t\le T.$

or

(b) $|f(x)| \le L(1+\|x\|^\lambda),\ |g(t,x)|\le L(1+\|x\|^\lambda) \qquad \forall x\in\mathbb{R}^d,\ 0\le t\le T$

with $L,\lambda\ge 1$.

(11.21) Theorem: Assume (11.18)-(11.20), (10.15), (10.16) and suppose that

(11.22) $$X_s^{(t,x)} = x + \int_t^s b(u, X_u^{(t,x)})\,du + \int_t^s \sigma(u, X_u^{(t,x)})\,dB_u \qquad t\le s<\infty$$

has a weak solution for every pair $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, which is unique in law. Let $v(t,x):[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ be of class $C^{1,2}$ and solves the Cauchy problem:

(11.23) $$-\frac{\partial v}{\partial t} + kv = L_t v + g \qquad \text{in } [0,T) \times \mathbb{R}^d$$
$$v(T,x) = f(x) \qquad x \in \mathbb{R}^d.$$

and satisfies for a $M>0$, $m \geq 1$.

(11.24) $$\max_{0\leq t\leq T} |v(t,x)| \leq M(1+|x|^{2m}) \qquad x \in \mathbb{R}^d$$

Then

(11.25) $$v(t,x) = E^{t,x}\Big[f(X_T)\exp\{-\int_t^T k(u,X_u)\,du\} + \int_t^T g(s,X_s)\exp\{-\int_t^s k(u,X_u)\,du\}\,ds\Big].$$

on $[0,T]\times\mathbb{R}^d$. In particular, such solution is unique.

<u>Proof:</u> Let $T_n := \inf\{s \geq t : \|X_s\| \geq n\}$. Using Itô's formula on the process $v(s,X_s)\exp\{-\int_t^s k(u,X_u)\,du\} =: Z_s$ together with the fact that v solves (11.23) we obtain

(11.26) $$\begin{aligned} v(t,x) = {} & E^{t,x}\Big[\int_0^{T\wedge T_n} g(s,X_s)\exp\{-\int_t^s k(\theta,X_\theta)\,d\theta\}\,ds\Big] \\ & + E^{t,x}\Big[v(T_n,X_{T_n})\exp\{-\int_t^{T_n} k(\theta,X_\theta)\,d\theta\}\,\mathbb{1}\{T_n\leq T\}\Big] \\ & + E^{t,x}\Big[f(X_T)\exp\{-\int_t^T k(\theta,X_\theta)\,d\theta\}\,\mathbb{1}\{T_n > T\}\Big]. \end{aligned}$$

<u>Exercise:</u> Prove (11.26). Use the generalisation of Itô's rule for explicitly time-dependent functions: For $f \in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^d,\mathbb{R})$ and semimartingale $X_t = X_0 + A_t + M_t$

$$\begin{aligned} f(t,X_t) = {} & f(0,X_0) + \int_0^t \partial_t f(s,X_s)\,ds + \sum_{i=1}^d \int_0^t \partial_i f(s,X_s)\,dA_s^{(i)} \\ & + \sum_{i=1}^d \int_0^t \partial_i f(s,X_s)\,dM_s^{(i)} \\ & + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \int_0^t \partial_{ij}^2 f(s,X_s)\,d\langle M^i,M^j\rangle_s \qquad 0\leq t<\infty. \end{aligned}$$

(11.27) <u>Exercise:</u> Prove that the solution to SDE (11.22) satisfies

(11.28) the estimate $E^{t,x}\big[\max_{t\leq u\leq s}\|X_u\|^{2m}\big] \leq C e^{C(s-t)}(1+|x|^{2m})$.

for $t\leq s\leq T$, $m\geq 1$ and $C = C(m,K,T,d) > 0$.

Using this we see that the first term in (11.26) converges as $n\nearrow\infty$ to

$$E^{t,x}\int_0^T g(s,X_s)\exp\{-\int_0^s k(\theta,X_\theta)\,d\theta\}\,ds,$$

Chapter 13: SEMIMARTINGALE LOCAL TIME

We will develop the concept of local time of continuous semimartingales with values in $\mathbb{R}$. Among others, this concept will allow us to understand more the Tanaka's equation (10.40) and to prove the claim that "$\int_0^t \operatorname{sign} X_s \, dX_s$ is adapted to $\mathcal{F}^{|X|}$" which appears after (10.43).

Intuition: For $(B_t)_{t\geq 0}$ a Brownian motion, we are interested "in the time that (B_t) spends at zero". One way to measure this time is to consider Lebesgue measure of the set $Z_t = \{s \leq t;\ B_s = 0\}$. However it can be shown that

(13.1) $$P\text{-a.s.} \quad \text{Lebesgue}(Z_\infty) = 0.$$

Other way is to consider formal expression

(13.2) $$l_t = \int_0^t \delta(B_s)\, ds$$

for δ being the Dirac δ-function at 0. As for $f(x) = |x|$ we have $f'(x) = \operatorname{sign} x$ and $\frac{1}{2} f''(x) = \delta(x)$, Itô's formula formally gives

(13.3) $$|B_t| = |B_0| + \int_0^t \operatorname{sign} B_s \, dB_s + l_t.$$

We are now going to prove (13.3) and its generalisation for an arbitrary continuous semimartingale.

(13.4) Theorem (Tanaka's formula) $((\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ as always$)$
Let X be a continuous semimartingale. Then there exists a continuous increasing adapted process $(l_t)_{t\geq 0}$ such that

(13.5) $$|X_t| - |X_0| = \int_0^t \operatorname{sign}(X_s)\, dX_s + l_t \qquad \leftarrow \text{Tanaka's formula}$$

where $\operatorname{sign}(x) = \pm 1$ depending on $x \leq 0$ or $x > 0$. (i.e. $\operatorname{sign} 0 = -1$)
The process l_t is called semimartingale local time of X at 0.
It grows only when $X = 0$:

(13.6) $$\int_0^t \mathbb{1}\{X_s \neq 0\}\, dl_s = 0$$

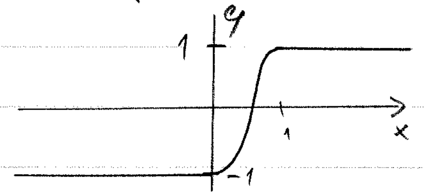
(13.7) Corollary: For X as above

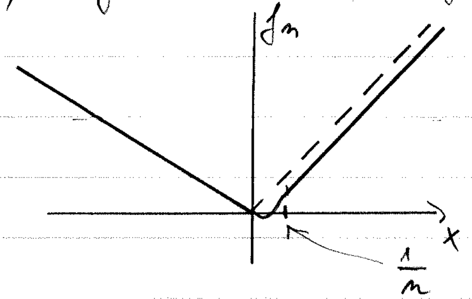
$$X_t^+ = X_0^+ + \int_0^t 1\!\!1\{X_s > 0\}\, dX_s + \tfrac{1}{2} \ell_t$$
$$X_t^- = X_0^- - \int_0^t 1\!\!1\{X_s \le 0\}\, dX_s + \tfrac{1}{2} \ell_t .$$

Indeed, this follows from $X_t = X_0 + \int_0^t dX_s$ and (13.5) using $X_t^+ - X_t^- = X_t$, $X_t^+ + X_t^- = |X_t|$.

Proof of (13.4): We approximate the function $f(x) = |x|$ by a C^2 function. Let $\varphi \in C^\infty$ be increasing $\varphi: \mathbb{R} \to [-1,1]$ such that $\varphi(x) = 1$ for $x \le 0$, $\varphi(x) = 1$ for $x \ge 1$. Define C^∞ functions f_n by

$$f_n(x) = -x \quad \text{on } x \le 0, \qquad f_n'(x) = \varphi(nx) \quad \text{for all } x \in \mathbb{R}.$$





Then, $f_n \xrightarrow{n\to\infty} f$ in $(C, \|\cdot\|_\infty)$ and $f_n'(x) \nearrow \operatorname{sign}(x)$.
By Itô's formula

(13.8) $$f_n(X_t) - f_n(X_0) = \int_0^t f_n'(X_s)\, dX_s + \underbrace{\tfrac{1}{2} \int_0^t f''(X_s)\, d\langle X\rangle_s}_{:= C_t^n}$$

As $f_n'' \ge 0$, C_t^n is continuous increasing process. From $f''(x) = 0$ for $|x| \ge n^{-1}$, we have

(13.9) $$\int_0^t 1\!\!1\{|X_s| > \tfrac{1}{n}\}\, dC_s^n = 0.$$

Let $X = X_0 + M + A$ be the canonical decomposition of X. By localisation, we may assume that M is bounded and A of bounded variation.
Then,

(13.10) $$\Big\| \int_0^\infty (\operatorname{sign} X_s - f_n'(X_s))\, dM_s \Big\|_{L^2(P)}^2 = E \int_0^\infty (\operatorname{sign} X_s - f_n'(X_s))^2\, d\langle M\rangle_s \longrightarrow 0 \quad \text{as } n\to\infty$$

By Doob's inequality

(13.11) $$\sup_t \Big| \int_0^t (\operatorname{sign} X_s - f_n'(X_s))\, dM_s \Big| \xrightarrow{n\to\infty} 0 \quad \text{in } L^2(P)$$

and by passing to a subsequence also P-a.s.

Also,

(13.12) $$\left| \int_0^t (\operatorname{sign} X_s - f_n'(X_s))\, dA_s \right| \le \int_0^t |\operatorname{sign} X_s - f_n'(X_s)|\, |dA_s|$$
$$= \int_0^t (\operatorname{sign} X_s - f_n'(X_s))\, |dA_s| \xrightarrow{n\to\infty} 0$$

by monotone convergence, uniformly in t, P-a.s.

Hence the LHS of (13.8) converges uniformly P-a.s to LHS of (13.5), and the stoch. integral on the RHS of (13.8) converges P-a.s. uniformly to the stoch. integral on the RHS of (13.5). This implies that C^n converges uniformly P-a.s. to a limit l which is continuous and increasing and (13.5) holds.

Viewing C^n as a distribution function of a measure on $\mathbb{R}_+$, we see that P-a.s. this measure converges vaguely to the measure associated to l (vague convergence means $\int f\, d\mu_n \to \int f\, d\mu$ for every $f \in C_c(\mathbb{R})$). This and (13.9) then imply

$$\int_0^t 1\!\!1\{|X_s| > \tfrac{1}{n}\}\, dl_s = 0$$

and (13.6) follow by the monotone convergence. $\square$.

(13.13) <u>Corollary</u> $\quad l_t = \int_0^t 1\!\!1\{X_s = 0\}\, dl_s \quad$ (follows directly from (13.9)).

We now come back to Tanaka's equation

(13.14). $$dX_t = \operatorname{sign} X_t\, dB_t, \quad X_0 = 0.$$

As on pages 113-114 we take $\Omega = (C(\mathbb{R}_+,\mathbb{R}), \mathcal{F})$, W - Wiener measure, X - canonical BM, $\mathcal{G}_t = \bar{\mathcal{F}}_t$ and set

(13.15) $$B_t = \int_0^t \operatorname{sign} X_s\, dX_s. \qquad \text{(c.f. (10.46))}$$

From Tanaka's formula we see that

$$B_t = |X_t| - l_t.$$

To show that (B_t) is $\mathcal{F}_t^{|X|}$-adapted we need:

(13.16) <u>Lemma:</u> Let X be BM, l_t^X its local time and $l_t^{|X|}$ the local time of $|X|$. Then

(13.17) $$l_t^{|X|} = 2 l_t^X$$

and, in particular, l_t^X is $\mathcal{F}^{|X|}$-adapted.

Proof: From Tanakas formula it follows that $|X|$ is a semimartingale, so $(\ell_t^{|X|})_{t\geq 0}$ is well defined. By Tanaka's formula

(13.18) $$d|X|_t = \operatorname{sign} X_t \, dX_t + d\ell_t^X$$

By definition of $\ell_t^{|X|}$, using $\operatorname{sign}(0) = -1$

(13.19) $$\begin{aligned} |X_t| - |X_0| &= \int_0^t \operatorname{sign}|X_s| \, d|X|_s + \ell_t^{|X|} \\ &= \int_0^t \left[1 - 2 1\{|X_s| = 0\}\right] d|X|_s + \ell_t^{|X|} \\ &\overset{(13.18)}{=} |X_t| - |X_0| - 2\int_0^t 1\{|X_s| = 0\} \operatorname{sign} X_s \, dX_s \\ &\qquad - 2\int_0^t 1\{|X_s| = 0\} \, d\ell_s^X + \ell_t^{|X|} \\ &\overset{(13.13)}{=} |X_t| - |X_0| - 0 - 2\ell_t^X + \ell_t^{|X|} \qquad P\text{-a.s.} \end{aligned}$$

where to see that the first integral equals zero we used $\| \int_0^t 1\{|X_s| = 0\} \operatorname{sign} X_s \, dX_s \|^2_{L^2(P)} = \int_0^t 1\{X_s = 0\} \, ds = 0$ by (13.1).
(13.17) then follows directly from (13.19). The second claim of the lemma then follows from (13.17). □.

(13.20) Remark: To see that (13.1) holds, it suffices to write

$$W[\operatorname{Leb}(Z_t)] = W\left[\int_0^t 1\{Z_t\} \, ds\right] = W\left[\int_0^t 1\{X_s = 0\} \, ds\right] \overset{\text{Fubini}}{=} \int_0^t W[X_s = 0] \, ds = 0.$$

Local time at x, continuity, occupation formula

Of course, zero does not play any particular role. For $a \in \mathbb{R}$ we may define the local time of X at a by

(13.21) $$\ell_t^a = |X_t - a| - |X_0 - a| - \int_0^t \operatorname{sign}(X_s - a) \, dX_s.$$

We are now going to show that the two-parameter process $(\ell_t^a, a \in \mathbb{R}, t \geq 0)$ has a "nice" version.

(13.22) <u>Theorem:</u> Assume that X is a local martingale. Then there exists a version of $\{l_t^a : a\in\mathbb{R}, t\ge 0\}$ which is jointly continuous, i.e.

(13.23) $$\lim_{\substack{s\to t\\ b\to a}} l_s^b = l_a^t \qquad \forall (t,a)\in\mathbb{R}_+\times\mathbb{R}$$

In addition, for any $\varphi:\mathbb{R}\to\mathbb{R}$ bounded measurable

(13.24) $$\int_0^t \varphi(X_s)\, d\langle X\rangle_s = \int_{\mathbb{R}} \varphi(a)\, l_t^a\, da.$$

(13.25) <u>Remark:</u> This theorem is an extension of famous Trotter's theorem which deals with X being a BM. In this case the occupation formula (13.24) reads

(13.26) $$\int_0^t \varphi(B_s)\, ds = \int_{\mathbb{R}} \varphi(a)\, l_t^a\, da.$$

which gives rigorous version of formula (13.2). Indeed, writing formally

(13.27) $$l_t^a = \int_0^t \delta_a(B_s)\, ds$$

then by Fubini theorem (formally)

(13.28) $$\int_{\mathbb{R}} \varphi(a) l_t^a\, da = \int_{\mathbb{R}}\int_0^t \varphi(a)\delta_a(B_s)\, ds\, da = \int_0^t\int_{\mathbb{R}} \varphi(a)\delta_a(B_s)\, da\, ds = \int_0^t \varphi(B_s)\, ds$$

<u>Proof:</u> $-$<u>(13.23)</u>. By localisation we may assume that $X, \langle X\rangle$ are bounded by a constant K. As

(13.29) $$l_t^a = |X_t - a| - |X_0 - a| - \int_0^t \operatorname{sign}(X_s - a)\, dX_s$$

and first two terms are jointly continuous in (a,t), we should find a jointly continuous version of the stoch. integral. Here again we use Kolmogorov's lemma (12.14), here with $d=1$, $S = (C(\mathbb{R}_+,\mathbb{R}), \|\cdot\|_\infty)$, and $\xi(a) = (t\mapsto \int_0^t \operatorname{sign}(X_s-a)\, dX_s)$.

Denoting $\xi(a,t) = \int_0^t \operatorname{sign}(X_s - a)\, dX_s$, we have for $a<b$, $p\ge 4$.

(13.30) $$E\Big[\sup_t |\xi(a,t)-\xi(b,t)|^p\Big] = 2^p E\Big[\sup_t \Big|\int_0^t 1\!\!1_{\{a<X_s\le b\}}\, dX_s\Big|^p\Big] \le C_p E\Big[\Big|\int_0^\infty 1\!\!1_{\{a<X_s\le b\}}\, d\langle X\rangle_s\Big|^{p/2}\Big],$$

by the Burkholder-Davis-Gundy inequality (7.83). To estimate the RHS, let $f\in C^2$ be such that $f'(x)=0$ for $x\le -1$ and f'' is as on the figure:

This implies $0 \le f'(x) \le 2$ for all $x \in \mathbb{R}$. Setting $\delta = b - a$ and $\varphi(x) = f(\delta^{-1}(x-a))$ we have

$$0 \le \frac{1}{2}\int_0^t 1\!\!1_{\{a < X_s \le b\}} \, d\langle X\rangle_s \le \frac{\delta^2}{2}\int_0^t \varphi''(X_s)\, d\langle X\rangle_s$$

$$\overset{\text{Itô}}{=} \delta^2 \Big\{ \varphi(X_t) - \varphi(X_0) - \int_0^t \varphi'(X_s)\, dX_s \Big\}$$

(13.31) $$\le \delta^2 |\varphi(X_t) - \varphi(X_0)| + \delta^2 \Big|\int_0^t \varphi'(X_s)\, dX_s\Big|$$

$$\le 2\delta |X_t - X_0| + \delta \Big| \int_0^t f'(\delta^{-1}(X_s - a))\, dX_s \Big|$$

$$\le 2K\delta + \delta \Big| \int_0^t f'(\delta^{-1}(X_s - a))\, dX_s \Big|$$

Hence, by BDG inequality (7.83) again,

$$E\Big[\Big|\int_0^\infty 1\!\!1_{\{a < X_s \le b\}}\, d\langle X\rangle_s\Big|^{p/2}\Big] \le$$

$$\le C_{p,K}\, \delta^{p/2} \Big(1 + E\Big[\Big|\int_0^\infty f'(\delta^{-1}(X_s - a))\, dX_s\Big|^{p/2}\Big]\Big)$$

(13.32) $$\le C_{p,K}\, \delta^{p/2} \Big(1 + E\Big[\Big|\int_0^\infty f'(\delta^{-1}(X_s - a))^2\, d\langle X\rangle_s\Big|^{p/4}\Big]\Big)$$

$$\le C_{p,K}\, \delta^{p/2} \Big(1 + 2^{p/2} E(\langle X\rangle_\infty^{p/4})\Big).$$

As we assume that $\langle X\rangle_\infty \le K$ we can assemble (13.30)–(13.32):

$$E\Big[\sup_t |\xi(a,t) - \xi(b,t)|^p\Big] \le A_{K,p} |a-b|^{p/2}$$

and as $p > 2$ Kolmogorov's lemma implies the existence of a continuous version of ℓ_t^a.

– (13.24) To prove this we use the following generalisation of Itô's formula dealing with convex (not necessarily C^2) functions. Recall that f is convex if for $x < y < z$

(13.33) $$(z-x) f(y) \le (z-y) f(x) + (y-x) f(z) \iff \frac{f(z)-f(x)}{z-x} \le \frac{f(z)-f(y)}{z-y}$$

It follows that

(13.34) $$D_- f(z) = \lim_{y \nearrow z} \frac{f(z)-f(y)}{z-y} \quad \text{exists}$$

Further, $\int_a^b (D_- f)(y)\, dy = f(b) - f(a)$ by MCT, $D_- f$ is left-continuous, nondecreasing. We may thus define measure μ^f on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

(13.35) $$\mu^f[a,b) = (D_- f)(b) - (D_- f)(a)$$

(13.36) <u>Proposition:</u> (Meyer-Itô formula) Let X be local mart, $(\ell^a_t)_{a,t}$ its continuous local time and $f:\mathbb{R}\to\mathbb{R}$ convex. Then

(13.37) $$f(X_t)-f(X_0) = \int_0^t D_-f(X_s)\,dX_s + \frac{1}{2}\int \ell^a_t\,\mu^f(da)$$

We finish the proof of (13.24) first. Assume first that $\varphi\geq 0$, $\varphi\in C(\mathbb{R},\mathbb{R})$, and pick $f\in C^2$ such that $f''=\varphi$. Observe that in this case

(13.38) $$\int_{\mathbb{R}} g(a)\mu^f(da) = \int_{\mathbb{R}} g(a)\varphi(a)\,da \qquad \text{for any } g\geq 0 \text{ measurable.}$$

Taking $g(a)=\ell^a_t$ and comparing (13.37) with the usual Itô's formula, (13.24) follows for $\varphi\geq 0$, $\varphi\in C$. For φ measurable one proceeds by usual arguments. □

<u>Proof of Proposition (13.36)</u>: By localisation we assume that X, $\langle X\rangle$ are bounded by K. Due to (13.6) we may then assume that μ^f is supported on $[-K,K]$. By scalling we can further consider only f with $\mu^f(\mathbb{R})=1$. Finally observing that if (13.37) is true for some f, then it holds also for $f(x)+cx$ for any c, we may assume that $D_-f=0$ on $(-\infty,-K]$.

If $\mu^f=\delta_a$ for a $a\in\mathbb{R}$, then (13.37) is essentially Tanaka's formula (because then $f=\frac{1}{2}|x-a|$, $D_-f=\frac{1}{2}\operatorname{sign}(x-a)$.). So (13.37) holds in this case. By linearity it then holds if the support of μ^f is a finite set.

For general μ^f, let Y be a r.v. with distribution μ, F its distribution function (i.e. F is right continuous version of D_-f). Set $h_n:\mathbb{R}\to\mathbb{R}$

(13.39) $$h_n(x)=j2^{-n} \quad \text{if} \quad (j-1)2^{-n}<x\leq j2^{-n}$$

and $Y_n=h_n(Y)$. Let F_n be dist. function of Y_n and μ_n its distribution. As supp $\mu^f\subset[-K,K]$, support of μ_n is finite, and linear extension of Tanakas formula yields

(13.40) $$f_n(X_t)-f_n(X_0) = \int_0^t D_-f_n(X_s)\,dX_s + \frac{1}{2}\int_{\mathbb{R}}\ell^a_t\,\mu_n(da).$$

where

(13.41) $$f_n(x) = \int_{-\infty}^{x} F_n(y)\,dy.$$

It is easy to see that

(a) $E[\varphi(Y_n)] = \int \varphi(a)\mu_n(da) \xrightarrow{n\to\infty} E\varphi(Y) = \int \varphi(a)\mu(da) \quad \forall \varphi \in C^b$

(13.42) (b) $F_n(x-) = D_- f_n(x) \nearrow F(x-) = D_- f(x)$

(c) $f_n(x) \nearrow f(x) = \int_{-\infty}^{x} F(y)\,dy.$

Using (a) and continuity of ℓ^a_t, $\int \ell^a_t \mu_n(da) \longrightarrow \int \ell^a_t \mu(da)$.
By standard L^2 argument, MCT and (b) we get that

$$\int_0^t D_- f_n(X_s)\,dX_s \xrightarrow{n\to\infty} \int_0^t D_- f(X_s)\,dX_s \text{ uniformly in } t \text{ in } L^2(P).$$

Finally (c) gives $f_n(X_t) - f_n(X_0) \longrightarrow f(X_t) - f(X_0)$ and the proposition is proved □

(13.42a) <u>Corollary</u>: Let X be a semimartingale. Then there exists a version of $\{\ell^a_t : a \in \mathbb{R}, t \in \mathbb{R}_+\}$ which is continuous in time and right continuous with left limits in space jointly.

$$\lim_{\substack{s\to t\\ b\searrow a}} \ell^b_s = \ell^a_t \quad ; \quad \lim_{\substack{s\to t\\ b\nearrow a}} = \ell^{a-}_t \text{ exists.}$$

Moreover, when $X = X_0 + M + A$ is the canonical decomposition of X,

$$\ell^a_t - \ell^{a-}_t = 2\int_0^t \mathbb{1}\{X_s = a\}\,dA_s.$$

Also, the occupation formula (13.24) holds.

<u>Proof</u>: Rewriting (13.29) in this setting we have

$$\ell^a_t = |X_t - a| - |X_0 - a| - \int_0^t \text{sign}(X_s - a)\,dM_s - \int_0^t \text{sign}(X_s - a)\,dA_s.$$

As the first three terms on the RHS possess jointly continuous version, we should only observe that, (as sign $0 = -1$)

(13.42b) $$\int_0^t \text{sign}(X_s - a)\,dA_s \text{ is cont. in } t \text{ and cadlag in } a.$$

Moreover,

$$\lim_{\substack{s\to t\\ b\nearrow a}} \int_0^t \text{sign}(X_s - a)\,dA_s = \int_0^t (1 - 2\mathbb{1}\{X_s < a\})\,dA_s = \int_0^t \text{sign}(X_s - (a-))\,dA_s$$

and thus.

$$\int \text{sign}(X_s - a)\,dA_s - \int \text{sign}(X_s - (a-))\,dA_s = -2\int \mathbb{1}\{X_s = a\}\,dA_s.$$

The proof of occupation formula above remains unchanged. □