Sheet 11

Exercises for May 23

Question 37. Counting lattice animals (Level B, 2 pts)
Consider a graph $G = (V, E)$ such that every vertex has degree at most D. A finite connected subset of G is called a lattice animal. For $x \in V$ and $n \in \mathbb{N}$, let $a(x, n)$ be the number of lattice animals with n vertices.

(a) Using percolation show that

$$\sum_{n\geq 1}[p(1-p)]^{Dn}a(n,x) \leq 1.$$

(b) Deduce that for every x and n, $a(n,x) \leq 4^{Dn}$, as used in the lecture.

(c) Show that one can replace $p^D(1-p)^D$ by $p(1-p)^D$ in the previous bound.

Question 38. Critical susceptibility (Level B, 3 pts)
Recall that $\phi_p(S) = p\sum_{xy\in\Delta S} P_p(0 \overset{S}{\leftrightarrow} x)$, for $S \subset \mathbb{Z}^d$.

(a) Show that the set $\{p \in [0,1] : \exists 0 \in S \subset \mathbb{Z}^d \text{ finite s.t. } \phi_p(S) < 1\}$ is open in $[0,1]$.

(b) Deduce that this set coincides with $[0, p_c)$, and thus $\phi_{p_c}(B_n) \geq 1$ for all $n \in \mathbb{N}$.

(c) Deduce from this that $E_{p_c}[|\mathcal{C}_0|] = \infty$.

Question 39. Correlation lenght (Level B, 5 pts)
Let $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$.

(a) Show that for any $p \in [0,1]$, $n, m \in \mathbb{N}$,

$$P_p(0 \leftrightarrow (n+m)e_1) \geq P_p(0 \leftrightarrow ne_1)P_p(0 \leftrightarrow me_1)$$

(b) Use this and subadditivity to show that

$$\xi(p) := \Big(\lim_{n\to\infty} -\frac{1}{n} \log P_p(0 \leftrightarrow ne_1) \Big)^{-1}$$

exists and that $P_p(0 \leftrightarrow ne_1) \leq \exp(-n/\xi(p))$. $\xi(p)$ is called correlation length.

(c) Use the BK inequality to show that

$$P_p(0 \leftrightarrow \partial B_{n+m}) \leq |\partial B_n| P_p(0 \leftrightarrow \partial B_n) P_p(0 \leftrightarrow \partial B_m).$$

(d) Deduce that for every n and $x \in \partial B_n$,

$$P_p(0 \leftrightarrow \partial B_n) \geq e^{-n/\xi_p}/(cn^{d-1}).$$

Hint. Observe that defining $\beta(n) = \log P_p(0 \leftrightarrow \partial B_n)$ and $g(n) = \log |\partial B_n|$, one has $\beta(m+n) \leq \beta(m) + \beta(n) + g(n)$. Use this to show that there is a constant c such that $a(n) = \beta(n) + g(n) + c$ is subadditive.

(e) Use the last exercise to show that $\xi_{p_c} = \infty$.