Sheet 6

Exercises for April 11

This time the questions should be easier. Question 20 can be handed in one week later, if you prefer.

Question 18. Ladder time of a random walk (Level B, 8 pts)
Let $(S_n)_{n\geq 0}$ be a random walk with $S_0 = 0$, and whose increments $X_n = S_n - S_{n-1}$ are i.i.d., take values in $\{-1, 0, 1, 2, \dots\}$, and satisfy $EX_i = 0$, $\operatorname{Var} X_i = \sigma^2 \in (0, \infty)$. We define the stopping time T, called the first (weak) ascending ladder time, by

$$T = \inf\{k \geq 1 : S_k \geq 0\}.$$

(a) Show that T is a.s. finite. *Hint:* What can you say about recurrence resp. transience of S?

(b) Show that, for every $k \geq 0$,

$$P[S_T = k] = P[X_1 = k] + P[X_1 = -1]P[S_T = k + 1 | S_T > 0].$$

Hint. Decompose the event $\{S_T = k\}$ according to value of X_1. In the case $X_1 = -1$, what is the distribution of the number of excursions from (-1) to (-1) staying strictly under (-1), before reaching $\{0, 1, \dots\}$ again?

(c) Use (b) to show that for any $k \geq 0$

$$P[S_T = k] = \sum_{j\geq 0} \Big(\frac{P[X_1 = -1]}{P[S_T > 0]}\Big)^j P[X_1 = k + j].$$

(d) Independently from the previous computations observe that

$$\sum_{k\geq 0} \sum_{j\geq k} P[X_1 = j] = 1.$$

Hint: Recall the formula $EY = \sum_{j\geq 1} P[Y \geq j]$ which holds for every $\{0, 1, \dots\}$-valued random variable Y.

(e) Combine (c) and (d) to show that $P[S_T > 0] = P[X_1 = -1]$.

(f) Deduce the final result of this exercise:

$$E[S_T] = \sigma^2/2.$$

Question 19. Ladder time sequence and claim (4.88) (Level B, 3 pts)
For the random walk as in the previous exercise, let $T_0 = 0$, and

$$T_i = \inf\{k > T_{i-1} : S_k = M_k\},$$

where $M_k = \sum_{i\leq k} S_i$ is the running maximum. Observe that $T_1 = T$. Let further $J_n := \#\{1 \leq i \leq n : S_i = M_i\}$.

(a) Observe $J_n = \sup\{k \geq 1 : T_k \leq n\}$.

(b) Deduce,

$$M_n = \sum_{k=1}^{J_n} (S_{T_k} - S_{T_{k-1}}).$$

(c) Show that

$$\lim_{n\to\infty} \frac{M_n}{J_n} = \frac{\sigma^2}{2}, \qquad \text{a.s.}$$

Question 20. Contour function of geometric GW process (Level A, 3 pts)
Let T_n be the critical GW tree with geometric offspring distribution (that is $P[X = k] = 2^{-(k+1)}$, $k \in \{0, 1, \dots\}$) conditioned on $|T_n| = n$. Show that its contour function C^n is a SRW conditioned on hitting 0 for the first time at instant $2(n-1)$. Deduce that, as $n \to \infty$ C^n, converges to a standard Brownian excursion, after a correct rescaling.