Sheet 5

Exercises for April 4

Question 15. Proof of Theorem 4.24 (Level B, 3 pts)
Complete the proof of Theorem 4.24 of the lecture notes by showing Claim 4.32. You may use all claims proved in the lecture.

Question 16. Multiplicative coalescent (Level B, 4 pts)
Recall from the lecture the definition of the multiplicative coalescent: It is a Markov process $X = (X(t))_{t\geq 0}$ taking values in the space $\ell^2_\downarrow$ of ordered non-negative ℓ^2-sequences $x = (x_1, x_2, \dots)$ (i.e. sequences satisfying $x_1 \geq x_2 \geq \cdots \geq 0$ and $\|x\|_2^2 := \sum_{i=1}^\infty x_1^2 < \infty$). To construct it, we consider a family $(\xi_{ij})_{i<j}$ of i.i.d. standard exponential random variables. Given initial condition $X(0) = x \in \ell^2_\downarrow$, and a time $t \geq 0$, we construct a graph $G(x,t)$ with vertex set $\mathbb{N}$, by connecting every pair of vertices (i,j) satisfying $\xi_{ij} \leq t x_i x_j$. Let $C^1(t), C^2(t), \dots$ be the connected components of $G(x,t)$ ordered by they weights, that is so that $\sum_{i\in C^k(t)} x_i \geq \sum_{i\in C^l(t)} x_i$ for every $k > l$. Finally, define $X(t) = (\sum_{i\in C^k(t)} x_i)_{k\geq 1}$, that is in $X(t)$ all points in one component of $G(x,t)$ are "merged into one point". We write P_x for the law of the process started from $X(0) = x$.

(a) Assume that $\|x\|_1 = \sum_{i\geq 1} x_i < \infty$. Show that, P_x-a.s., for every $t \geq 0$, $G(x,t)$ contains only finitely many edges. *Hint:* Compute and estimate the expected number of edges in $G(x,t)$.

(b) Show that if $\|x\|_1 = \infty$, then P_x-a.s, for every $t > 0$, every point i has infinitely many neighbours in $G(x,t)$. *Hint:* Second Borel-Cantelli Lemma.

(c) Observe that $t \mapsto S(t) = \|X(t)\|_2^2$ is non-decreasing, i.e. it is not a priori clear that $X(t) \in \ell^2_\downarrow$ for all $t \geq 0$.

(d) For $x \in \ell^2_\downarrow$ and $k \in \mathbb{N}$, define $x^k = (x_1, \dots, x_k, 0, 0, 0)$ (truncated sequence). Couple now P_x and P_{x^k} by using the same ξ_{ij}'s, and write $S(x,t)$, $S(x^k,t)$ for the process S started from x or x^k, respectively. Show that, a.s., as $k \to \infty$

$$S(x^k, t) \nearrow S(x,t) \in [0, \infty].$$

Question 17. Mulitiplicative coalescent - continued (Level C, 4 pts)
We now show that $X(t) \in \ell^2_\downarrow$ for all $t \geq 0$, P_x-a.s.

Let C be the space of bounded continuous functions from $\ell^2_\downarrow$ to $\mathbb{R}$. For $x \in \ell^2_\downarrow$, let x^{i+j} be the element of $\ell^2_\downarrow$ obtained by merging x_i and x_j, that is x^{i+j} is the ordered version of the sequence $(x_i + x_j, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-i}, x_{j+1}, \ldots)$. For $f \in C$ and $x \in \ell^2_\downarrow$ define

$$(Af)(x) = \sum_i \sum_{j>i} x_i x_j \big(f(x^{i+j}) - f(x)\big)$$

(a) Assume that $\|x\|_1 < \infty$. Show that

$$(Af)(x) = \lim_{t \to 0} \frac{1}{t} \big\{ E_x[f(X(t))] - f(x) \big\},$$

that is A is the generator of X (restricted to ℓ^1).

(b) What can you say about the process

$$M_f(t) = f(X(t)) - \int_0^\infty (Af)(X(t))\, \mathrm{d}t.$$

(c) Assume $\|x\|_1 < \infty$ and let S be as in the previous question. Show that under P_x, the process $Y(t) = t + \frac{1}{S(t)}$ is a submartingale.

(d) Use (c) to show the inequality

$$P_x(S(t) \geq s) \leq \frac{tsS(0)}{s - S(0)}$$

Hint. Bound $E_x[\frac{1}{S(t)}]$ from above by $\frac{1}{s} + E_x\big[(\frac{1}{S(t)} - \frac{1}{s})_+\big]$ and use that $S(t)$ is non-decreasing.

(e) Use (d) of the last question to show that the inequality in (c) holds for all $x \in \ell^2_\downarrow$. Then use it to show that $X(t) \in \ell^2_\downarrow$, P_x-a.s.