Sheet 4

Exercises for March 28

Question 11. Connectivity and expected cluster size (Level B, 4 pts)
Consider the ER graph with parameters n and $p = \lambda/n$.

(a) Prove that the expected cluster size of a given vertex, $\chi(\lambda) = E_{n,\lambda/n}[|\mathcal{C}(1)|]$, satisfies

$$\chi(\lambda) = 1 + (n-1)P_{n,\lambda/n}(1 \leftrightarrow 2).$$

(b) Show that

$$P_{n,\lambda/n}(1 \leftrightarrow 2) = \zeta_\lambda^2(1 + o(1)). \tag{1}$$

Hint. Use the results of the lecture, in particular the "No middle ground" lemma.

Question 12. Second largest critical cluster (Level B-C, 4 pts)
Consider the ER graph again. For $\lambda > 1$, show that the second largest component $\mathcal{C}_{(2)}$ of the ER graph satisfies

$$\frac{|\mathcal{C}_{(2)}|}{\log n} \xrightarrow[n\to\infty]{P_{n,\lambda/n}} \frac{1}{I_{\mu_\lambda}}.$$

Hint. Observe first that conditionally on the giant component $\mathcal{C}_{\max}$, the states of the edges not incident to the giant component are independent Bernoulli(λ/n) random variables. Hence, conditionally on $|\mathcal{C}_{\max}| = m$, the $ER(n, \lambda/n)$-graph with the giant component removed has the same distribution as $ER(n-m, \lambda/n)$ graph. Finally, observe that for $m \sim \zeta_\lambda n$, i.e. $n - m \sim \eta_\lambda n$, one has

$$\frac{\lambda}{n} \sim \frac{\lambda\eta_\lambda}{n-m} = \frac{\mu_\lambda}{n-m}.$$

Question 13. Critical case trivia (Level A, 2 pts)
Consider now the critical ER graph with parameters n and $p = n^{-1} + \theta n^{-4/3}$. Show that Lemma 4.4 of the lecture implies that $E_{n,p}[|\mathcal{C}(1)|] \geq cn^{1/3}$, for some $c > 0$. That is, the upper bound of Lemma 4.5 is of correct order.