Chapter A: CONSTRUCTION OF PROCESSES

Stochastic process in continuous time is a family of random variables $(X_t)_{t\in T}$ constructed on some common probability space $(\Omega, \mathcal{A}, P)$, indexed by "time" t in $T = \mathbb{R}_+$ or $T = \mathbb{R}$.

It is natural to ask if such a family exists, or, more precisely, can be constructed. One frequently used construction is the following:

Let T be an arbitrary non-empty index set. Our goal is to construct a family $(X_t)_{t\in T}$ of E-valued random variables, assuming that we are given so-called finite dimensional marginals, i.e. for every $J \subset T$ finite we are given the distribution Q_J of the vector $(X_t)_{t\in J}$.

We will assume that $(E, \mathcal{E})$ nice Polish measurable space

In our construction we will take

(A.1) $$\Omega = E^T,$$

the space of all functions from T to E, and use "canonical" coordinates as our process:

(A.2) $$X_t(\omega) = \omega(t) \qquad \text{for} \quad \omega = \{t \mapsto \omega(t)\} \in \Omega.$$

As σ-algebra on Ω, we choose the smallest σ-algebra making all X_t measurable

(A.3). $$\mathcal{A} = \sigma(X_t, t \in T)$$

This σ-algebra is the standard product σ-algebra on E^T, generated by cylinder sets

(A.4). $$\left\{ \prod_{t\in J} A_t \times \prod_{s\in T\setminus J} E \;:\; J \subset T \text{ finite},\ A_t \in \mathcal{E} \right\}.$$

We also write

(A.5) $$\Omega_J = \prod_{t\in J} E = E^J, \quad \mathcal{A}_J = \bigotimes_{t\in J} \mathcal{E} \qquad J\subset T$$

and for $T\supset J\supset K$ we write $\pi_{J,K}: \Omega_J \to \Omega_K$ for the canonical projection

(A.6) $$\pi_{J,K}(\omega) = \{\omega(t): t\in K\} \qquad \text{for } \omega = \{J\ni t\mapsto \omega(t)\}$$

We set

$$\pi_K := \pi_{T,K}.$$

With this notation Q_J can be viewed as a measure on $(\Omega_J, \mathcal{A}_J)$. And our goal is to find a measure Q on $(\Omega, \mathcal{A})$ such that Q_J are its finite-dimensional marginals, that is

(A.7) $$(\pi_J)_\# Q = Q \qquad \text{for all } J\subset T \text{ finite}$$

It is clear that for Q to exist, Q_J's cannot be completely arbitrary: Since obviously for $K\subset J\subset T$

$$(\pi_{J,K})_\# ((\pi_J)_\# Q) = (\pi_K)_\# Q$$

Q_J's must satisfy:

(A.8) $$(\pi_{J,K})_\# Q_J = Q_K \qquad \forall\, K\subset J\subset T \text{ finite.}$$

The following theorem, which is an important tool to construct stochastic processes, shows that this is roughly what is needed:

(A.9) __Theorem__ (Kolmogorov extension theorem).
Let E be a Polish space and $\mathcal{E}$ its Borel σ-field and Q_J's satisfy (A.8). Then there exists a unique probability measure Q on $(\Omega, \mathcal{A})$ such that (A.7) holds.

__Proof:__ See literature.

Markov Processes

- another important class of processes
- not really covered extensively in the lecture
- introduced to provide another view on SDE'S (Itô).

(1) **Def.** Let $(E, \mathcal{E})$ be a measurable space. A transition kernel on E is a map $Q: E \times \mathcal{E} \to [0,1]$ satisfying
(a) $\forall x \in E$, $\mathcal{E} \ni A \mapsto Q(x,A)$ is a probability measure
(b) $\forall A \in \mathcal{A}$, $E \ni x \mapsto Q(x,A)$ is $\mathcal{E}$-measurable.

(2) **Example:** E-countable, Q is determined by transition "matrix" $Q_{xy} = Q(x, \{y\})$.

For $f \in \mathcal{B}(E) = \{f: E \to \mathbb{R}$ bounded measurable$\}$ we set

(3) $$Qf(x) = \int_E Q(x,dy) f(y) \in \mathcal{B}(E).$$

(4) **Def:** A collection $(Q_t)_{t \geq 0}$ of transition kernels on E is called transition semigroup if
(i) $\forall x \in E$ $\quad Q_0(x,dy) = \delta_x(dy)$
(ii) $\forall s,t \geq 0, A \in \mathcal{E}$,

(5) $$Q_{t+s}(x,A) = \int_E Q_t(x,dy)\, Q_s(y,A).$$

(Chapman-Kolmogorov)

(iii) $\forall A \in \mathcal{E}$ $\quad (t,x) \mapsto Q_t(x,A)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}$-meas.

Set $\|f\| = \sup\{|f(x)| : x \in E\}$ a norm on $\mathcal{B}(E)$. It is easy to see that $\|Q_t f\| \leq \|f\|$. Also (5) can be viewed as $Q_{s+t} = Q_s Q_t$, i.e. $(Q_t)_t$ is a semigroup of contractions on E.

(6) Def Let $(\Omega, \mathcal{A}, (\mathcal{A}_t), P)$ be a probability space and $(Q_t)_{t\geq 0}$ a transition semigroup on E. A Markov process (w.r.t. $(\mathcal{A}_t)$) with trans. sg. (Q_t) is an $\mathcal{A}_t$-adapted process $(X_t)_{t\geq 0}$ with values in $\underline{E}$ s.t. for $\forall s,t \geq 0$, $f \in B(E)$

(7) $$E[f(X_{s+t}) \mid \mathcal{A}_s] = Q_t f(X_s) \qquad P\text{-a.s.}$$

(8) Rem. • If the filtration is not specified, we take the canonical one $\mathcal{A}_t^X = \sigma(X_s : s \leq t)$.

• (7) implies $P[X_{s+t} \in A \mid \mathcal{A}_s] = Q(X_s, A)$.

• By easy induction argument, denoting γ law of X_0

$$P(X_0 \in A_0, \dots, X_{t_n} \in A_n) =$$

(7) $$= \int_{A_0} \gamma(dx_0) \int_{A_1} Q_{t_1}(x_0, dx_1) \int_{A_2} Q_{t_2-t_1}(x_1, dx_2) \dots \int_{A_n} Q_{t_n-t_{n-1}}(x_{t_{n-1}}, dx_n)$$

$\forall A_i \in \mathcal{E}$, $n \geq 0$, $0 = t_0 < t_1 \dots < t_n$.

or, for $\forall f_i \in B(E)$

(8) $$E[f_0(X_0) \dots f_n(X_n)] = \\ = \int \gamma(dx_0) f_0(x_0) \int Q_{t_1-t_0}(x_0, dx_1) f_1(x_1) \dots \int Q_{t_n-t_{n-1}}(x_{n-1}, dx_n) f_n(x_n).$$

• If (8) holds for some collection (Q_t) of trans. kernels, then X_t is a M.P.

(9) Example: $Q_t(x, dy) = \frac{1}{\sqrt{2\pi t}^{\,d}} e^{-\frac{|x-y|^2}{2t}}$

X is (pre-) BM.

Existence of Markov processes:

Kolmogorov extension theorem on separate sheet.

(10) <u>Corollary</u>: Let $(E, \mathcal{E})$ be Polish as above, and (Q_t) a transition semigroup on E. Let γ be a prob. on $(E, \mathcal{E})$. Then there exists a unique probability P on $\Omega = E^{\mathbb{R}_+}$, $\mathcal{A} = \mathcal{E}^{\otimes \mathbb{R}_+}$ under which the canonical process $(X_t)_{t \geq 0}$ is a Markov process with transition semigroup (Q_t) and $X_0 \sim \gamma$.

<u>Proof</u>: Set $J = \{t_1, \dots, t_n\}$ with $t_1 < \dots < t_n$ and define Q_J via

$$Q_J(A) = \int \gamma(dx_0) \dots \int Q_{t_n - t_{n-1}}(x_{n-1}, dx_n) \, 1\!\!1_A(x_1, \dots, x_n)$$

for $A \in \mathcal{E}^{\otimes J}$. (5) can be easily used to show the consistency of Q_J's. So Kolm. theorem gives existence of (X_t). Rem (8), its last point then yields the fact that X is Markov.

(11) <u>Def</u>: For $x \in E$, we set P_x the measure from the above corollary when $\gamma = \delta_x$. $x \mapsto P_x$ is measurable.

(12) <u>Rem</u>: The process constructed in (10) can have "very irregular" trajectories.

The following is an important concept:

(13) <u>Def</u>: Let $\lambda > 0$. The λ-resolvent of the trans. semigroup (Q_t) is the linear operator $R_\lambda : B(E) \to B(E)$ given by

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} Q_t f(x) \, dt \quad , f \in B(E), x \in E.$$

(Here we need (4(iii)) to ensure that $t \mapsto Q_t f(x)$ is meas.).

(14) <u>Lemma</u>: Properties of the resolvent

(i) $\|R_\lambda f\| \le \frac{1}{\lambda}\|f\|$

(ii) If $0 \le f \le 1$, then $0 \le \lambda R_\lambda f \le 1$.

(iii) (Resolvent equation) If $\lambda, \mu > 0$, then

$$R_\lambda - R_\mu + (\lambda-\mu) R_\lambda R_\mu = 0.$$

<u>Proof</u>: (i), (ii) – Exercise.

(iii) $$R_\lambda (R_\mu f)(x) = \int_0^\infty e^{-\lambda s} Q_s \left(\int e^{-\mu t} Q_t f \, dt \right)(x) \, ds$$

$$= \int_0^\infty e^{-\lambda s} \left(\int Q_s(x,dy) \int_0^\infty e^{-\mu t} Q_t f(y) \, dt \right) ds$$

$$= \int_0^\infty e^{-\lambda s} \left(\int_0^\infty e^{-\mu t} Q_{s+t} f(y) \, dt \right) ds$$

$$= \int_0^\infty e^{-(\lambda-\mu)s} \left(\int_0^\infty e^{-\mu(s+t)} Q_{s+t} f(y) \, dt \right) ds$$

$$= \int_0^\infty e^{-(\lambda-\mu)s} \left(\int_s^\infty e^{-\mu r} Q_r f(y) \, dr \right) ds$$

$$= \int_0^\infty Q_r f(y) e^{-\mu r} \left(\int_0^r e^{-(\lambda-\mu)s} \, ds \right) dr$$

$$= \int_0^\infty Q_r f(y) \left(\frac{e^{-\mu r} - e^{-\lambda r}}{\lambda - \mu} \right) dr$$

$$= \frac{1}{(\lambda-\mu)} \left(R_\mu f - R_\lambda f \right) \qquad \square.$$

(15) <u>Example</u>: For BM

$$R_\lambda f(x) = \int r_\lambda(y-x) f(y) \, dy \qquad \text{with}$$

$$r_\lambda(y-x) = \int_0^\infty e^{-\lambda t} p_t(y-x) \, dt = \frac{1}{\sqrt{2\lambda}} e^{-|y-x|\sqrt{2\lambda}}.$$

(16) <u>Lemma</u> Let (X_t) be a M.P. with S.g. (Q_t) on $(\Omega, \mathcal{A}, (\mathcal{A}_t), P)$. Then, for every $h \in B(E)$, non-negative, and $\lambda > 0$,

$$Y_t = e^{-\lambda t} R_\lambda h(X_t)$$

is $\mathcal{A}_t$ - supermartingale.

<u>Proof</u>: Y_t is bdd and thus in L^1. Moreover, with $s \geq 0$,

$$e^{-\lambda s} Q_s R_\lambda h = e^{-\lambda s} \int_0^\infty e^{-\lambda t} Q_{s+t} h \, dt = \int_s e^{-\lambda r} Q_r h \, dr \leq R_\lambda h.$$

and thus, for $s, t \geq 0$

$$E[Y_{s+t} \mid \mathcal{A}_t] = E[e^{-\lambda(s+t)} R_\lambda h(X_{s+t}) \mid \mathcal{A}_t] =$$

$$= Q_s(e^{-\lambda(t+s)} R_\lambda h(X_t)) \leq e^{-\lambda t} R_\lambda h(X_t) = Y_t. \quad \square$$

<u>Feller semigroups</u>

We assume that E is metrisable locally compact & countable at infinity, i.e. E is a countable union of compact set, and $\mathcal{E}$ is its Borel σ-algebra. Then E is Polish & there is a increasing sequence K_n of compact sets s.t. $\bigcup K_n = E$.

A function $f: E \to E$ tends to 0 at infinity if $\forall \varepsilon > 0$ $\exists K_\varepsilon$ compact s.t. $|f| \leq \varepsilon$ on $E \setminus K_\varepsilon$. Equivalently

$$\lim_{n \to \infty} \sup_{x \in E \setminus K_n} |f(x)| = 0.$$

We define

$$C_0(E) : \{ f : E \to \mathbb{R},\ f \text{ is continuous and tends to 0 at } \infty \}$$

It is a Banach space with norm $\|f\| = \sup_{x \in E} |f(x)|$.

(17) Def: Let $(Q_t)_{t\geq 0}$ be a transition semigroup. It is called Feller semigroup if

(a) $\forall f \in C_0(E)\ \forall t \geq 0 \qquad Q_t f \in C_0(E)$

(b) $\forall f \in C_0(E) \quad \lim_{t\to 0} \| Q_t f - f \| = 0.$

A Markov process in E is a Feller process if its transition semigroup is Feller.

(18) Rem: (a) One can show that (b) can be replaced by

$$\forall f \in C_0(E)\ \forall x \in E \quad (Q_t f)(x) \xrightarrow{t\to 0} f(x).$$

(b) (b) implies that for every $s \geq 0$

$$\lim_{t\to 0} \| Q_{s+t} f - Q_s f \| = \lim \| Q_s (Q_t f - f) \| = 0,$$

as Q_s is contraction on $B(E)$ and thus $C_0(E)$

So the limit is actually uniform in s, i.e. $t \mapsto Q_t f$ is unif. continuous.

We now fix $(Q_t)_{t\geq 0}$ a Feller semigroup. Then, by dominated

(19) convergence, $R_\lambda f \in C_0(E) \qquad \forall f \in C_0(E), \lambda > 0.$

(20) Lemma: Let $\lambda > 0$ and set $\mathcal{R} = \{ R_\lambda f : f \in C_0(E) \}$. Then $\mathcal{R}$ does not depend on λ and $\mathcal{R}$ is a dense subspace of $C_0(E)$

Proof: For $\lambda \neq \mu$, the resolvent equation gives

$$R_\lambda f = R_\mu (f + (\mu - \lambda) R_\lambda f)$$

It follows that $\{ R_\lambda f : f \in C_0(E) \} \subset \{ R_\mu f : f \in C_0(E) \}$, opposite inclusion follows by $\lambda \leftrightarrow \mu$.

$\mathcal{R}$ is clearly linear subspace of $C_0(E)$ and

$$\lambda R_\lambda f = \lambda \int_0^\infty e^{-\lambda t} Q_t f = \int_0^\infty e^{-t} Q_{t/\lambda} f \xrightarrow[\lambda\to\infty]{} f \text{ in } C_0(E)$$

by (17(b)) and DCT. $\square$

(21) <u>Definition</u>: Let
$$D(L) = \left\{ f \in C_0(E) : \frac{Q_t f - f}{t} \text{ converges in } C_0(E) \text{ as } t \downarrow 0 \right\}$$
and for $f \in D(L)$ define
$$Lf = \lim_{t \downarrow 0} \frac{Q_t f - f}{t}.$$
Then $D(L)$ is a linear subspace of $C_0(E)$, $L : D(L) \to C_0(E)$ is a linear operator, called generator of $(Q_t)_{t \geq 0}$; $D(L)$ is called domain of L.

(22) <u>Lemma</u>: Let $f \in D(L)$ and $s \geq 0$. Then $Q_s f \in D(L)$ and
$$L(Q_s f) = Q_s(Lf).$$

<u>Proof</u>: Obviously
$$\frac{Q_t(Q_s f) - Q_s f}{t} = Q_s\left(\frac{Q_t f - f}{t} \right).$$

If $f \in D(L)$ and since Q_s is contraction, the RHS converges as $t \to 0$, and thus $(Q_s f) \in D(L)$.

(23) <u>Lemma</u>: If $f \in D(L)$, then
$$Q_t f = f + \int_0^t Q_s(Lf)\, ds = f + \int_0^t L(Q_s f)\, ds.$$

<u>Proof</u>: For every $t \geq 0$,
$$\frac{1}{\varepsilon}(Q_{t+\varepsilon} f - Q_t f) = Q_t\left(\frac{1}{\varepsilon}(Q_\varepsilon f - f) \right) \xrightarrow[\varepsilon \to 0]{} Q_t(Lf)$$
uniformly as t ranges over $\mathbb{R}_+$. Hence $t \mapsto (Q_t f)(x)$ is differentiable on $\mathbb{R}_+$ with derivative $Q_t(Lf)(x)$ which is also continuous. Hence, the first equality follows. The second comes from (22).

The next Proposition connects resolvent & generator

(24) Proposition: ($\lambda > 0$)

(i) If $g \in C_0(E)$, then $R_\lambda g \in D(L)$ and $(\lambda - L) R_\lambda g = g$.

(ii) If $f \in D(L)$, then $R_\lambda (\lambda - L) f = f$.

As consequence, $D(L) = \mathcal{R}$ and the operators $R_\lambda : C_0(E) \to \mathcal{R}$ and $\lambda - L : D(L) \to C_0(E)$ are the inverse of each other.

Proof: (i) For $g \in C_0(E)$, $\varepsilon > 0$

$$\varepsilon^{-1}(Q_\varepsilon R_\lambda g - R_\lambda g) = \varepsilon^{-1}\left(\int_0^\infty e^{-\lambda t} Q_{\varepsilon+t}\, g\, dt - \int e^{-\lambda t} Q_t g\, dt\right)$$

$$= \varepsilon^{-1}\left((1 - e^{-\lambda\varepsilon})\int_0^\infty e^{-\lambda t} Q_{\varepsilon+t}\, g\, dt - \int_0^\varepsilon e^{-\lambda t} Q_t g\, dt\right)$$

$$\xrightarrow{\varepsilon \to 0} \lambda R_\lambda g - g,$$

using (17(A)) and DCT. This implies $R_\lambda g \in D(L)$ and $L(R_\lambda g) = \lambda R_\lambda g - g$, proving the claim.

(ii) For $f \in D(L)$ we know by (23), $Q_t f = f + \int_0^t Q_s(Lf)\, ds$

Therefore, $\int_0^\infty e^{-\lambda t} Q_t f(x)\, dt = \frac{f(x)}{\lambda} + \int_0^\infty e^{-\lambda t}\left(\int_0^t Q_s(Lf)(x)\, ds\right) dt$

$$= \frac{f(x)}{\lambda} + \int_0^\infty \frac{e^{-\lambda s}}{\lambda} Q_s(Lf)(x)\, ds$$

That is $\lambda R_\lambda f = f + R_\lambda(Lf)$, yielding (ii).

The last claim follows from (i), (ii).

(i) $\Rightarrow$ $\mathcal{R} \subset D(L)$, (ii) $\Rightarrow$ $D(L) \subset \mathcal{R}$, (i)+(ii) $\Rightarrow$ "inversion claim".

(25) Corollary L (together with $D(L)$) determines $(Q_t)_{t \geq 0}$.

Proof: Let $f \in C_0(E)$, $f \geq 0$. Then $R_\lambda f$ is the unique element of $D(L)$ such that $(\lambda - L) R_\lambda f = f$. In addition, knowing $R_\lambda f(x) = \int e^{-\lambda t} Q_t f(x)\, dt$ for every $\lambda > 0$ determines the continuous function $t \mapsto Q_t f(x)$. But Q_t is determined by its values on non-negative functions.

(26) <u>Example:</u> (Brownian motion). It is easy to see that the semigroup of BM is Feller. We know that

$$R_\lambda f(x) = \int \frac{1}{\sqrt{2\lambda}} \exp\{-\sqrt{2\lambda}\,|y-x|\} f(y)\,dy.$$

If $h \in D(L)$, then $h = R_\lambda f$ for some $f \in C_0(\mathbb{R})$. With $\lambda = \frac{1}{2}$

$$h(x) = \int \exp\{-|y-x|\} f(y)\,dy.$$

Taking derivatives (justify the computation)

$$h'(x) = \int \operatorname{sgn}(y-x) \exp\{-|y-x|\} f(y)\,dy$$

and for $x > x_0 \in \mathbb{R}$

$$\begin{aligned} h'(x) - h'(x_0) &= \int \left(\operatorname{sgn}(y-x) e^{-|y-x|} - \operatorname{sgn}(y-x_0) e^{-|y-x_0|} \right) f(y)\,dy \\ &= \int_{x_0}^{x} \left(-e^{-|y-x|} - e^{-|y-x_0|} \right) f(y)\,dy \\ &\quad + \int_{\mathbb{R}\setminus[x_0,x]} \operatorname{sgn}(y-x_0) \left(e^{-|y-x|} - e^{-|y-x_0|} \right) f(y)\,dy. \end{aligned}$$

Hence,

$$\frac{h'(x) - h'(x_0)}{x - x_0} \xrightarrow{x \downarrow x_0} -2f(x_0) + h(x_0)$$

and similarly for $x \uparrow x_0$. Hence h is twice-differentiable and $h'' = -2f + h$. Since $h = R_{1/2} f$, we know that

$$\left(\tfrac{1}{2} - L\right) h = f$$

hence $Lh = -f + \frac{1}{2}h = \frac{1}{2}h''$.

Hence, $D(L) \subset \{ h \in C^2(\mathbb{R}),\ h \,\&\, h'' \in C_0(\mathbb{R}) \}$

and $Lh = \frac{1}{2}h''$ for $h \in D(L)$ □

On the other hand, if $g \in C^2$ with $g, g'' \in C_0(\mathbb{R})$, we set $f = \frac{1}{2}(g - g'') \in C_0(\mathbb{R})$ and then $h = R_{1/2} f \in D(L)$. Hence, by the last argument, $h \in C^2$ and $h'' = -2f + h$. Therefore $(h-g)'' = h - g$. But since $h - g \in C_0(\mathbb{R})$, we have $h - g \equiv 0$, i.e. $g = h \in D(L)$.

<u>Rem:</u> Determining $D(L)$ is difficult in general.

Martingales related to Markov processes

Let $(Q_t)_{t\geq 0}$ be Feller semigroup, and for every $x \in E$, $(X_t^x)_{t\geq 0}$ be Markov with sg. (Q_t) s.t. $P(X_0^x = x) = 1$.
We assume that paths of X^x are cadlag (which is Ok).

(27) Thm: Let $h, g \in C_0(E)$. TFAE
(i) $h \in D(L)$ and $Lh = g$.
(ii) $\forall x \in E$

$$M_t^x = h(X_t^x) - \int_0^t g(X_s^x)\, ds$$

is a mart.

Proof: (i) $\Rightarrow$ (ii) If $g = Lh$, then by (23)

$$Q_s h = h + \int_0^s Q_r g \, dr$$

Hence, for $t \geq 0$, $s \geq 0$

$$E[h(X_{t+s}^x) \mid \mathcal{F}_t] = Q_s(h(X_t^x)) = h(X_t^x) + \int_0^s (Q_r g)(X_t^x)\, dr$$

and

$$E\left[\int_t^{t+s} g(X_r^x)\, dr \mid \mathcal{F}_t\right] = \int_t^{t+s} E[g(X_r^x) \mid \mathcal{F}_t]\, dr = \int_0^s Q_r g(X_t^x)\, dr.$$

Combining these two finishes the proof.
(ii) $\Rightarrow$ (i) If (ii) holds, then for every $t \geq 0$

$$E\left[h(X_t^x) - \int_0^t g(X_r^x)\, dr\right] = h(x)$$

but also $\qquad = Q_t h(x) - \int_0^t Q_r g(x)\, dr.$

Hence

$$\frac{Q_t h - h}{t} = \frac{1}{t}\int_0^t Q_r g \, dr \xrightarrow{t \to 0} g$$

in $C_0(E)$ as Q is Feller.