Sheet 10

Exercises for November 29

Question 34. Hitting probabilities (Level A, 2 pts)
Let W_x be the d-dimensional Wiener measure, $d \geq 2$, X the canonical process (i.e. a Brownian motion in $\mathbb{R}^d$ with $X_0 = x$), and let $K \subset \mathbb{R}^d$ compact. Prove that the function $x \mapsto W_x[H_K < \infty]$ is harmonic on $\mathbb{R}^d \setminus K$.

Hint. Use the strong Markov property and the fact that $f : U \to \mathbb{R}$ is harmonic on U iff $\int_{\partial B(y,r)} f(u) \mathrm{d}\sigma(u) = f(y)$ for every y, r such that $\overline{B(y,r)} \subset U$; here σ denotes the surface measure on the sphere $\partial B(y,r)$.

Question 35. Polar sets (Level B, 4 pts)
Assume now that K is *polar*, that is there exists a point $x \in \mathbb{R}^d$ such that $W_x[H_K < \infty] = 0$.

(a) Show that $\mathbb{R}^d \setminus K$ is connected and that $W_x[H_K < \infty] = 0$ for all $x \in \mathbb{R}^d \setminus K$.
Hint. Observe that the set $\{x \in \mathbb{R}^d \setminus K : W_x[H_K < \infty] = 0\}$ is both open and closed in $\mathbb{R}^d \setminus K$.

(b) If $\alpha \in (0, d]$, we say that K has zero α-dimensional Hausdorff measure if, for every $\varepsilon > 0$, we can find an integer N_ε and open balls $B(x_i, r_i)$, $i = 1, \ldots, N_\varepsilon$, such that $K \subset \bigcup_{i=1}^{N_\varepsilon} B(x_i, r_i)$, and
$$\sum_{i=1}^{N_\varepsilon} (r_i)^\alpha \leq \varepsilon.$$
Show that if $d \geq 2$ and K has zero $d-2$-dimensional Hausdorff measure, then K is polar.

Question 36. (Level B, 4 pts)
Consider now $d = 2$ and identify $\mathbb{R}^2$ with $\mathbb{C}$. For $\alpha \in (0, \pi)$ consider the open cone $\mathcal{C} = \{r e^{\mathrm{i}\theta} : r > 0, \theta \in (-\alpha, \alpha)\}$. Let $T = T_{\mathcal{C}}$ be the exit time from $\mathcal{C}$.

(a) Show that the law of $\log |X_T|$ under W_1 agrees with the law of $\beta_{\inf\{s \geq 0 : |\gamma_s| = \alpha\}}$, where β and γ are two independent 1-dimensional Brownian motions started from 0.

(b) Verify that for every $\lambda \in \mathbb{R}$,

$$W_1\left[e^{i\lambda \log |X_T|}\right] = \frac{1}{\cosh(\alpha\lambda)}.$$

Hint. Use that $z \mapsto e^{i\lambda z}$ is holomorphic, i.e. its real and imaginary part are harmonic.

Question 37. Liouville's theorem (Level B, 2 pts)
Show that every bounded function which is harmonic in whole plane $\mathbb{R}^2$ must be constant. Use recurrence of BM in $d = 2$.

Question 38. D'Alembert's theorem (Level B, 3 pts)
Let P be a non constant polynomial with complex coefficients. Use the properties of complex BM to show that, for any $\varepsilon > 0$, the compact set $\{z \in \mathbb{C} : |P(z)| \leq \varepsilon\}$ is non empty. Conclude that there exists a solution to $P(z) = 0$.