Sheet 9

Exercises for November 22

Question 30. Exponential martingales and SDE (Level A, 3 pts)
For any semimartingale X we define (similarly as in the lecture)

$$\mathcal{E}(\lambda X)_t = \exp\{\lambda X_t - \frac{\lambda^2}{2}\langle X\rangle_t\}.$$

Show the following identity:

$$\mathcal{E}(\lambda X)_t = \mathcal{E}(\lambda X)_0 + \int_0^t \lambda \mathcal{E}(\lambda X)_s \, \mathrm{d}X_s.$$

This can be interpreted as: $\mathcal{E}(\lambda X)$ is a solution to the stochastic differential equation (therefore the terminology)

$$\mathrm{d}Y_t = \lambda Y_t \, \mathrm{d}X_t.$$

Question 31. Uniquenes of the solution (Level B, 3 pts)
In the setting of the previous exercise, assuming $X_0 = 0$, show that $\mathcal{E}(X)$ is the unique solution to the given SDE with $Y_0 = 1$ and $\lambda = 1$.

Hint. If Y is another solution, compute $Y\mathcal{E}(\lambda X)^{-1}$ using the integration by parts formula. To this end, show that $\mathrm{d}(\mathcal{E}(X)^{-1}) = -\mathcal{E}(X)^{-2}\mathrm{d}\mathcal{E}(X) + \mathcal{E}(X)^{-3}\mathrm{d}\langle\mathcal{E}(X)\rangle$ (which should be interpreted as above and proved using Ito formula). Observe also that Question 30 implies that $\mathrm{d}\langle\mathcal{E}(X)\rangle = \mathcal{E}(X)^2\mathrm{d}\langle X\rangle$.

Question 32. Exponential of the sum (Level A, 2 pts)
For two semimartingales X, Y, compute $\mathcal{E}(X+Y)$. Compare it to $\mathcal{E}(X)\mathcal{E}(Y)$. When does the equality hold?

Question 33. (Level B, 4 pts)
Let B be a Brownian motion with $B_0 = 1$. Fix $\varepsilon \in (0,1)$ and set $H_\varepsilon = \inf\{t \geq 0 : B_t = \varepsilon\}$. We also let $\lambda > 0$ and $\alpha \in \mathbb{R}$.

(a) Show that $(B_t^{H_\varepsilon})^\alpha$ is a semimartingale and give its canonical decomposition into martingale and f.v. process.

(b) Show that

$$Z_t = (B_t^{H_\varepsilon})^\alpha \exp\Big\{ -\lambda \int_0^{t\wedge H_\varepsilon} \frac{\mathrm{d}s}{B_s^2} \Big\}$$

is a continuous local martingale if α and λ satisfy certain polynomial equation which you should determine.

(c) Compute

$$E\Big[\exp\Big\{ -\lambda \int_0^{H_\varepsilon} \frac{\mathrm{d}s}{B_s^2} \Big\} \Big].$$