Sheet 5

Exercises for October 25

The purpose of this sheet is to recall/introduce the concept of uniform integrability and to develop further the discrete martingale theory.

The sheet probably requires more work than usual. It will be compensated the next week.

Question 16. Martingale a.s. convergence (discrete time) (Level A, 3 pts)
Let $(X_n)_{n\geq 0}$ be a submartingale with $\sup_{n\geq 0} E[X_n^+] < \infty$. Show that there exists a random variable X_∞ such that $\lim_{n\to\infty} X_n = X_\infty$ a.s.

Hint: With help of the discrete-time upcrossing inequality show that $M_{a,b}^X(\mathbb{N}) < \infty$ a.s. for every $a < b \in \mathbb{Q}$. To this end use that, by monotone convergence, $E[M_{a,b}^X(\mathbb{N})] = \lim_{n\to\infty} E[M_{a,b}^X(\{0,\dots,n\})]$.

Question 17. Martingale L^p-convergence (discrete time) (Level B, 3 pts)
Let $p \in (1,\infty)$ and $(X_n)_{n\geq 0}$ be a martingale with $\sup_{n\geq 0} E[|X_n|^p] < \infty$. Show that there is X_∞ such that $\lim_{n\to\infty} X_n = X_\infty$ a.s. and also in L^p.

Hint. Use the previous exercise to show that X_n converges a.s. Use then Doob's L^p-inequality to show that $(\sup_{n\geq 0} |X_n|) \in L^p$ and apply the dominated convergence to deduce the result.

Question 18. Uniform integrability, L^1-convergence (Level -, 0 pts)
Read the following notes on Uniform integrability:

The previous exercise leaves open the case $p = 1$. Actually, the statement does not hold for $p = 1$, in general. On the other hand, if $\sup_n E[|X_n|] < \infty$, then, by the first exercise, X_n converges a.s. We also know one way how to deduce L^1 convergence from the a.s. convergence, namely the dominated convergence theorem, whose application requires the existence of L^1-dominating function.

We develop here another condition allowing to deduce L^1-convergence from the a.s. one. We will see that this condition is not only sufficient, but also necessary. It deals with general families of random variables, not only with martingales.

Definition. A collection $(X_i)_{i\in I}$ of random variables is said to be *uniformly integrable* (UI) if

$$\lim_{M\to\infty} \sup_{i\in I} E\big[|X_i| \mathbf{1}\{|X_i| > M\}\big] = 0.$$

Example. (a) When $|X_i| < Y$ for all $i \in I$ and some $Y \in L^1$, that is there is L^1-dominating function, then $(X_i)_{i\in I}$ is UI. (Try to show that!)

(b) Let $\varphi \geq 0$ be a function such that $\lim_{x\to\infty} \varphi(x)/x = \infty$. Examples are $\varphi(x) = x^p$, $p > 1$, or $\varphi(x) = x\log^+(x)$. If $\sup_{i\in I} E\varphi(|X_i|) < \infty$, then $(X_i)_{i\in I}$ is UI.

To see that this is true, set $A = \sup_i E\varphi(|X_i|)$ and choose $\varepsilon > 0$ and $M < \infty$ such that $\inf_{u\geq M} \frac{\varphi(u)}{u} \geq \frac{A}{\varepsilon}$. Then for all $i \in I$,

$$\begin{aligned} E\big[|X_i|\mathbf{1}\{|X_i| > M\}\big] &\leq \frac{\varepsilon}{A} E\Big[\frac{\varphi(|X_i|)}{|X_i|}|X_i|\mathbf{1}\{|X_i| > M\}\Big] \\ &\leq \frac{\varepsilon}{A} E[\varphi(|X_i|)] \leq \varepsilon, \end{aligned}$$

which implies the condition of the definition of UI. ◊

Lemma. *Let $X \in L^1(\Omega, \mathcal{A}, P)$. Then the family*

$$\{E[X|\mathcal{G}] : \mathcal{G} \subset \mathcal{A} \text{ is a } \sigma\text{-algebra}\}$$

is UI.

Proof. We start with a technical claim:

Claim. *If $X \in L^1$ then for every $\varepsilon > 0$ exists $\delta > 0$ such that*

$$P[A] < \delta \quad \Longrightarrow \quad E[|X|; A] \leq \varepsilon.$$

Proof. Assume, by contradiction, that there is a sequence of events A_n with $P[A_n] \leq \frac{1}{n}$ and $E[|X|; A_n] \geq \varepsilon$. It follows that $|X|\mathbf{1}\{A_n\} \to 0$ in probability and thus a.s. along a subsequence k_n. For such sub-sequence, the dominated convergence theorem implies $E[|X|; A_{k_n}] \xrightarrow{n\to\infty} 0$, leading to contradiction. □

Fix now ε and δ as in the claim and choose $M < \infty$ such that $E|X|/M \leq \delta$. For $\mathcal{G} \subset \mathcal{A}$, by Jensen's inequality

$$\begin{aligned} E\big[|E[X|\mathcal{G}]|; |E[X|\mathcal{G}]| \geq M\big] &\leq E\Big[E\big[|X|\big|\mathcal{G}\big]; \underbrace{E\big[|X|\big|\mathcal{G}\big] \geq M}_{\in\mathcal{G}}\Big] \\ &= E\big[|X|; E\big[|X|\big|\mathcal{G}\big] \geq M\big], \end{aligned} \tag{1}$$

where the equality follows from the definition of the conditional expectation. In addition, by Chebyshev's inequality

$$P\big[E\big[|X|\big|\mathcal{G}\big] \geq M\big] \leq M^{-1}E\big[E\big[|X|\big|\mathcal{G}\big]\big] = M^{-1}E[|X|] \leq \delta,$$

and thus, by the claim, the right-hand side of (1) is bounded by ε, proving the UI property. □

The following theorem explains the usefullness of the UI property for dealing with L^1-convergence.

Theorem. *If $X_n \xrightarrow{n\to\infty} X$ a.s (actually 'in probability' would be sufficient, but we restrict to the a.s. case), then the following are equivalent*

(i) $\{X_n : n \geq 1\}$ *is UI,*

(ii) $X_n \xrightarrow{n\to\infty} X$ *in* L^1,

(iii) $E|X_n| \xrightarrow{n\to\infty} E|X| < \infty$.

Proof. *(i)* $\Longrightarrow$ *(ii).* For $M > 0$,

$$\begin{aligned} E[|X_n - X|] \leq\ & E[|X_n - X|; |X_n| \leq M, |X| \leq M] \\ & + 3E[|X_n|; |X_n| > M] + 3E[|X|; |X| > M]. \end{aligned} \tag{2}$$

For $\varepsilon \in (0,1)$, (i) implies the existence of M_0 such that

$$\sup_n E[|X_n|; |X_n| \geq M] \leq \frac{\varepsilon}{2} \qquad \text{for all } M \geq M_0.$$

By Fatou's lemma

$$E[|X|] \leq \liminf E[|X_n|] \leq \frac{\varepsilon}{2} + M_0 \leq M_0 + 1.$$

Hence, we may choose M so that, uniformly in n, the last two terms in (2) are smaller than $\frac{\varepsilon}{2}$. Hence,

$$\limsup E|X_n - X| \leq \limsup E[|X_n - X|; |X_n| \leq M, |X| \leq M] + \varepsilon = \varepsilon$$

by the dominated convergence theorem. As ε is arbitrary, (ii) follows.

(ii) $\Longrightarrow$ *(iii).* By Jensen's inequality

$$\big|E|X_n| - E|X|\big| \leq E\big[\big||X_n| - |X|\big|\big] \leq E[|X_n - X|] \to 0$$

by (ii), which implies (iii).

(iii) $\Longrightarrow$ *(i).* Fix $\varepsilon > 0$. Let $\psi_M : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that

$$\psi_M(x) = \begin{cases} x & \text{if } x \leq M-1, \\ \text{linear} & \text{on } x \in [M-1, M], \\ 0 & \text{if } x \geq M. \end{cases}$$

By the dominated convergence theorem, for M large enough, $E|X| - E\psi_M(|X|) \leq \frac{\varepsilon}{2}$. Another application of the dominated convergence theorem implies that

$$E[\psi_M(|X_n|)] \xrightarrow{n\to\infty} E[\psi_M(|X|)], \tag{3}$$

so by (iii), for all n larger than some n_0

$$E[|X_n|; |X_n| \geq M] \leq E[|X_n|] - E\psi_M(|X_n|) \leq E[|X|] - E[\psi_M(|X|)] + \frac{\varepsilon}{2} < \varepsilon.$$

By increasing M, the last inequality is valid for all n's, that is $(X_n)_{n\geq 1}$ is UI. $\square$

As a corollary we obtain a L^1-convergence theorem for submartingales.

Theorem. *For a submartingale $(X_n)_{n\geq 0}$ the following are equivalent*

(i) $(X_n)_{n\geq 0}$ *is UI,*

(ii) $X_n \to X$ *in* L^1 *and* P*-a.s.*

(iii) $X_n \to X$ *in* L^1.

Proof. *(i)* $\Longrightarrow$ *(ii).* The UI property implies $\sup E|X_n| < \infty$, so by Question 16, $X_n \to X$, P-a.s. The previous theorem then implies that $X_n \to X$ in L^1.

(ii) $\Longrightarrow$ *(iii)* is obvious.

(iii) $\Longrightarrow$ *(i)* Since $X_n \to X$ in L^1, we have also $X_n \to X$ in probability. The claim then follows by another application of the previous theorem. (Actually, here the proof is not complete, we need $X_n \to X$ a.s. to apply the previous theorem. Since this implication is not so important, we leave its proof 'partly' open). $\square$

Theorem. *For a* $\mathcal{F}_n$*-martingale* $(X_n)_{n\geq 0}$ *the following are equivalent*

(i) $(X_n)_{n\geq 0}$ *is UI,*

(ii) $X_n \to X_\infty$ *in* L^1 *and* P*-a.s.*

(iii) $X_n \to X_\infty$ *in* L^1.

(iv) There is a random variable X *such that* $X_n = E[X|\mathcal{F}_n]$

Proof. *(i)*$\Leftrightarrow$*(ii)*$\Leftrightarrow$*(iii)* follows from the last theorem.

(iii) $\Longrightarrow$ *(iv).* Let $n < m$. Then, for every $A \in \mathcal{F}_n$ $E[X_n \mathbf{1}_A] = E[X_m \mathbf{1}_A] \xrightarrow[m\to\infty]{(iii)} E[X_\infty \mathbf{1}_A]$, that is $X_n = E[X_\infty|\mathcal{F}_n]$ for all $n \geq 0$.

(iv) $\Longrightarrow$ *(i)* is a direct consequence of the lemma above. $\square$

Question 19. Backward martingale convergence (Level B, 3 pts)

(a) Let $(X_n)_{n\le 0}$ be a martingale with respect to "filtration" $(\mathcal{A}_n)_{n\le 0}$. (Here all definitions are "as usual" only the index set is different, that is $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, X_n is $\mathcal{A}_n$ measurable, and $X_n = E[X_{n+1}|\mathcal{A}_n]$ for all $n \le -1$.). Show that there is a random variable X such that $X = \lim_{n\to-\infty} X_n$ a.s. and also in L^1.

Hint. Use the upcrossing inequality as before and recall also that $X_n = E[X_0|\mathcal{F}_n]$ for all $n < 0$. This allows you to apply the previous theory.

(b) Use this theorem to complete the proof of Theorem 4.18 of the notes in the case when X is a martingale. Namely, show that $X_{t_k} \xrightarrow{k\to\infty} X_{t+}$ in L^1, with the notation of the notes.