Sheet 4

Exercises for October 18

Level A exercises are not difficult and should be understood by everybody. Level B exercises are more difficult but doable. Level C exercises are optional, solving them is a bonus for the final exam.

Question 12. Right-continuous filtrations (Level A, 2 pts)
Show the following claims:

(a) Let $(\mathcal{A}_t)_{t\geq 0}$ be a right-continuous filtration. Then T is $\mathcal{A}_t$-stopping time iff $\{T < t\} \in \mathcal{A}_t$ for all $t \geq 0$.

(b) The filtration $(\mathcal{F}_t^+)_{t\geq 0}$ defined in the lecture is right-continuous.

Question 13. Properties of stopping times (Level A, 6 pts)
Let S, T be stopping times with respect to $(\mathcal{A}_t)_{t\geq 0}$. Show the following claims

(a) The collection $\mathcal{A}_T := \{A \in \mathcal{A} : A \cap \{T \leq t\} \in \mathcal{A}_t \forall t \geq 0\}$ is a σ-algebra.

(b) T is $\mathcal{A}_T$-measurable.

(c) $S \vee T$ and $S \wedge T$ are stopping times.

(d) If $S \leq T$, then $\mathcal{A}_S \subset \mathcal{A}_T$.

(e) Both $\{S < T\}$ and $\{S \leq T\}$ belongs to $\mathcal{A}_S \cup \mathcal{A}_T$.

(f) Let T_n be a monotone increasing sequence of stopping times. Then $T = \lim_{n\to\infty} T_n$ is a stopping time. What happens when T_n is monotone decreasing?

Question 14. Zero set of BM (Level B, 2 pts)
Let $Z = \{t \in [0, 1] : B_t = 0\}$. Show that Z is a.s. a compact subset of $[0, 1]$ with no isolated points and zero Lebesgue measure.

Hint. To show that Z has no isolated points, first use the results of the lecture to show that $0 \in Z$ is a.s. not an isolated point of Z. For $q \in \mathbb{Q}$, let $d_q = q + H_0 \circ \theta_q = \inf\{s > q : B_s = 0\}$ be the first zero point after q. Use the strong Markov property to show that the event

$$\bigcup_{q \in \mathbb{Q}} \{d_q \text{ is an isolated point of } Z\}$$

has zero probability. Use this to complete the proof.

Remark. Z can be viewed as a 'random analogue' of Cantor middle-third set.

Question 15. Arcsine law (Level C, 3 pts)
Let $T = \inf\{t \geq 0 : B_t = M_1\}$ where $M_t = \max_{s \leq t} B_s$.

(a) Show that $T < 1$ a.s. (*Hint:* See (3.56) and its proof.) and that T is not a stopping time.

(b) Show that T has so-called arcsine distribution whose density is

$$f_T(x) = \frac{1}{\pi\sqrt{t(1-t)}} \mathbf{1}_{(0,1)}(t).$$

Hint. Let $Y_t = M_t - B_t$. Observe that $\{T \leq t\} = \{\sup_{s \leq 1-t}(B_{t+s} - B_t) \leq Y_t\}$.

(c) Show that (a), (b) remain valid if T is replaced by $L := \sup\{t \leq 1 : B_t = 0\}$.

Remark. To understand the name "arcsine law" try to compute the distribution function of T.