Sheet 2

Exercises for October 4

Level A exercises are not difficult and should be understood by everybody. Level B exercises are more difficult but doable. Level C exercises are optional, solving them is a bonus for the final exam.

Question 4. Gaussian process characterisation of pre-BM (Level A, 2 pts)
Prove the converse to Proposition 3.2 of lecture notes: If $(X_t)_{t\geq 0}$ is a centred Gaussian process with covariance function $E[X_s X_t] = s \wedge t$, then X is a pre-Brownian motion.

Hint. Adapt the proof of Proposition 3.3.

Question 5. Invariances of pre-BM (Level A, 4 pts)
Prove Proposition 3.6 of lecture notes: Let $(B_t)_{t\geq 0}$ be a pre-Brownian motion. Then all the following processes are pre-Brownian motions as well:

(a) $-B_t$

(b) $B_t^\lambda := \lambda^{-1} B_{\lambda^2 t}$, for every $\lambda > 0$.

(c) $\beta_t := t B_{1/t}$, $t \geq 0$, $\beta_0 := 0$.

(d) $B_t^{(s)} = B_{t+s} - B_s$, for every $s \geq 0$.

Question 6. L^2-differentiability of Gaussian processes (Level B, 4 pts)
Let $(X_t)_{t\in[0,1]}$ be a Gaussian process with a covariance function Γ.

(a) Show that the mapping $t \mapsto X_t$ from $[0,1]$ to $L^2(\Omega)$ is continuous iff Γ is continuous on $[0,1]^2$.

(b) Assume now that Γ is twice continuously differentiable. Show that, for every $t \in [0,1]$ the limit

$$\dot{X}_t := \lim_{h\to 0} \frac{1}{h}(X_{t+h} - X_t)$$

exists in $L^2(\Omega)$. Verify that $(\dot{X}_t)_{t\in[0,1]}$ is a Gaussian process and compute its covariance function.

Question 7. Gaussian conditioning (Level B, 3 pts)
Let $X = (X_1, \dots, X_d)$[1] be a $\mathbb{R}^d$-valued centred Gaussian vector with a covariance matrix C and for $n < d$ make partitions

$$X = (X^1, X^2) = ((X_1, \dots, X_n), (X_{n+1}, \dots, X_d)),$$
$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}, \end{pmatrix}$$

where C_{11} is a $n \times n$-matrix. Assume that C_{22} is invertible and show that conditionally on $\sigma(X^2)$, X^1 is a Gaussian vector with mean $C_{12}C_{22}^{-1}X^2$ and covariance matrix $C_{11} - C_{12}C_{22}^{-1}C_{21}$

Hint. Let $T = C_{12}C_{22}^{-1}$ and set $W = X^1 - TX^2$. Show that $E[W(X^2)^T] = 0$ (as a matrix) and deduce the independence of W and X^2. Use it to compute $E[X^1|\sigma(X^2)]$ and then also the conditional covariance.

[1] Here, all vectors should be understood as column vectors, but for sake of compactness of presentation we use the row form.