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ON THE TIGHTNESS OF THE MAXIMUM OF BRANCHING BROWNIAN MOTION IN RANDOM ENVIRONMENT

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We consider one-dimensional branching Brownian motion in a spatially random branching environment (BBMRE) and show that for almost every realisation of the environment, the distributions of the maximal particle of the BBMRE recentred around its median are tight as time evolves. This result is in stark contrast to the fact that the transition fronts in the solution to the randomised Fisher–Kolmogorov–Petrovskii–Piskunov (F-KPP) equation are, in general, not bounded uniformly in time. In particular, this highlights that—when compared to the settings of homogeneous branching Brownian motion and the F-KPP equation in a homogeneous environment—the introduction of a random environment leads to a much more intricate behaviour.

1. Introduction. The behaviour of the position of the maximally—or, equivalently, minimally—displaced particle in various variants of branching random walk (BRW) and branching Brownian motion (BBM) has been the subject of intensive research over the last couple of decades [1, 3, 13, 14, 16, 30]. While initially most of the work focused on branching systems with homogeneous branching rates, there has recently been an increased activity in the investigation of branching random walks with nonhomogeneous branching rates that depend on either time or space mostly in special *deterministic* ways; see [7, 11, 12, 17, 23, 24, 29, 34, 35, 37, 38, 40, 41].

In this article, we continue the study of the maximally displaced particle in the model of branching Brownian motion with spatially random branching environment (BBMRE). The study of BBMRE was initiated in [20] as a tool for investigating properties of the randomised Fisher–Kolmogorov–Petrovskii–Piskunov (F-KPP) equation, building on the previous work [17] on a discrete-space analogue, that is, branching random walk in i.i.d. random environment (BRWRE). The techniques developed in [17, 20] also lend themselves to obtain refined information on the front of the solution of the randomised F-KPP equation [18]. Subsequently, the techniques and results of [17] have been extended to the continuum space setting of BBMRE in [29].

We complement the above body of research by addressing a seemingly simple, but subtle problem that arises naturally, and which has also been formulated as an open question in [17]. More precisely, we show that the distributions of the position of the maximally displaced particle of the BBMRE, when recentred around its median, form a tight family of distributions as time evolves. While establishing tightness might a priori not look like an overly intricate problem, we take the opportunity to emphasise that such a preconception is erroneous; see also [14, 15]. Our result is particularly interesting as it sharply contrasts the result established in [18], where it is shown that the transition fronts of the solution to the randomised F-KPP equation are, in general, unbounded in time. In the homogeneous setting, such a dichotomy

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cannot be observed since, a fortiori, there is a duality between these two objects in that tightness of the recentred maximum of BBM is equivalent to the uniform boundedness in time of the transition fronts of the solution to the F-KPP equation.

1.1. *Homogeneous BBM and F-KPP equation.* To explain this duality in more detail, we start by recalling the model in the homogeneous situation, which will also serve as a point of reference throughout the article. For a (binary) branching Brownian motion with homogeneous branching rate equal to one, started from a single particle located at the origin at time 0, we denote its maximal displacement at time $t \geq 0$ by $M(t)$, and write

$$w(t,x) = P\big(M(t) \geq x\big), \tag{1.1}$$

for the probability that this displacement exceeds $x \in \mathbb{R}$. Then the function $w(t,x)$ solves a nonlinear PDE, known as the Fisher–Kolmogorov–Petrovskii–Piskunov equation,

$$\partial_t w(t,x) = \frac{1}{2}\partial_x^2 w(t,x) + w(t,x)\big(1 - w(t,x)\big), \quad t > 0, x \in \mathbb{R}, \tag{1.2}$$

with the initial datum $w(0,\cdot) = \mathbf{1}_{(-\infty,0]}$ of Heaviside type; see [31, 42]. Moreover, it is well known that as $t \to \infty$, the solution to (1.2) approaches a *travelling wave* g in the following sense: for an appropriate function $m : [0,\infty) \to [0,\infty)$, one has that

$$w\big(t, m(t) + \cdot\big) \to g \quad \text{uniformly as } t \to \infty \tag{1.3}$$

for a decreasing function g satisfying $\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to-\infty} g(x) = 1$. A critical ingredient in the proof of this convergence is that, again for $m(t)$ being chosen appropriately, one has

$$\begin{aligned} &w\big(t, x + m(t)\big) \quad \text{is decreasing in } t \text{ for } x < 0, \text{ and} \\ &w\big(t, x + m(t)\big) \quad \text{is increasing in } t \text{ for } x > 0. \end{aligned} \tag{1.4}$$

Property (1.3) immediately yields for every $\varepsilon > 0$ the existence of some $r_\varepsilon \in (0,\infty)$ such that

$$w\big(t, m(t) + r_\varepsilon\big) - w\big(t, m(t) - r_\varepsilon\big) > 1 - \varepsilon \quad \text{for all } t \geq 0. \tag{1.5}$$

Put differently, the family $(M(t) - m(t))_{t\geq 0}$ is tight. Another, essentially trivial, consequence of (1.3) is the uniform boundedness of the width of the transition front of the solution to (1.2); that is, that for every $\varepsilon \in (0, 1/2)$,

$$\limsup_{t\to\infty} \operatorname{diam}\big(\{x \in \mathbb{R} : w(t,x) \in [\varepsilon, 1-\varepsilon]\}\big) < \infty. \tag{1.6}$$

In this context, it is worth pointing out that the above line of reasoning implicitly uses the reflection symmetry of Brownian motion and the homogeneity of the branching environment. As a consequence, this proof technique breaks down in the presence of an inhomogeneous environment, and the relationship between the solutions of the F-KPP equation and the maximum of BBMRE becomes more intricate than that given in (1.1) and (1.2); cf. Section 3.1.

1.2. *Randomised F-KPP equation.* In the inhomogeneous setting of a random potential, as considered in the current paper, the respective randomised F-KPP equation has been investigated in [18]. In that source it has been established that, for a canonical choice of random potentials ξ, the transition front of the solution to the inhomogeneous F-KPP equation (which is discussed in more detail in Section 3.1)

$$\partial_t w^\xi(t,x) = \frac{1}{2}\partial_x^2 w^\xi(t,x) + \xi(x) w^\xi(t,x)\big(1 - w^\xi(t,x)\big), \quad t > 0, x \in \mathbb{R}, \tag{1.7}$$

with the initial condition $w^\xi(0,\cdot) = \mathbf{1}_{(-\infty,0]}$ does not need to be uniformly bounded in time. More precisely, in contrast to (1.6), it follows from [18], Theorem 2.3, that there are random potentials ξ within the class of inhomogeneities considered in the current paper, such that $\mathbb{P}$-a.s., for all $\varepsilon \in (0, 1/2)$,

$$\limsup_{t\to\infty} \operatorname{diam}\big(\{x \in \mathbb{R} : w^\xi(t,x) \in [\varepsilon, 1-\varepsilon]\}\big) = +\infty. \tag{1.8}$$

It might hence be surprising and is nontrivial to prove that for BBMRE in the random potential ξ we obtain tightness for the recentred family of maxima, and a novel approach is required in order to address this situation adequately.

It is worthwhile to note that the PDE results of [18] have been obtained by taking advantage of almost exclusively probabilistic techniques. In the current article, however, the probabilistic main result will be proven via a combination of analytic and probabilistic techniques.

2. Definition of the model and the main result.

We work with a model of *branching Brownian motion in random branching environment* (BBMRE) introduced in [18, 20] as a continuous space version of the branching random walk in random environment model studied in [17]. The random environment is given by a stochastic process $\xi = (\xi(x))_{x\in\mathbb{R}}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which fulfils the following assumptions.

ASSUMPTION 1.

- ξ is stationary, that is, for every $h \in \mathbb{R}$ one has

$$\big(\xi(x)\big)_{x\in\mathbb{R}} \overset{(d)}{=} \big(\xi(x+h)\big)_{x\in\mathbb{R}}. \tag{2.1}$$

- ξ fulfils a ψ*-mixing* condition: There exists a continuous nonincreasing function $\psi : [0,\infty) \to [0,\infty)$ satisfying $\sum_{k=1}^\infty \psi(k) < \infty$ such that (using the notation $\mathcal{F}_A = \sigma(\xi(x) : x \in A)$ for $A \subset \mathbb{R}$) for all $Y \in L^1(\Omega, \mathcal{F}_{(-\infty,j]}, \mathbb{P})$, and all $Z \in L^1(\Omega, \mathcal{F}_{[k,\infty)}, \mathbb{P})$ we have

$$\begin{aligned} \big|\mathbb{E}\big[Y - \mathbb{E}[Y] | \mathcal{F}_{[k,\infty)}\big]\big| &\le \mathbb{E}\big[|Y|\big]\psi(k-j),\\ \big|\mathbb{E}\big[Z - \mathbb{E}[Z] | \mathcal{F}_{(-\infty,j]}\big]\big| &\le \mathbb{E}\big[|Z|\big]\psi(k-j). \end{aligned} \tag{2.2}$$

  (Note that these conditions imply the ergodicity of ξ with respect to the usual shift operator.)
- The sample paths of ξ are $\mathbb{P}$-a.s. locally Hölder continuous, that is, for almost every ξ there exists $\alpha = \alpha(\xi) \in (0,1)$ and for every compact $K \subseteq \mathbb{R}$ a constant $C = C(K,\xi) > 0$ such that

$$\big|\xi(x) - \xi(y)\big| \le C|x-y|^\alpha \quad \text{for all } x, y \in K. \tag{2.3}$$

- ξ is uniformly bounded in the sense that the essential infimum and supremum of the random variable $\xi(0)$ (and thus also of $\xi(x)$, $x \in \mathbb{R}$, by (2.1)) satisfy

$$0 < \mathtt{ei} := \operatorname{ess\,inf} \xi(0) < \operatorname{ess\,sup} \xi(0) =: \mathtt{es} < \infty. \tag{2.4}$$

In the current article, we do not explicitly make use of the mixing condition. However, in particular in Section 5, we will employ some of the results developed in [18, 20], which depend on this mixing assumption.

The dynamics of BBMRE started at a position $x \in \mathbb{R}$ is as follows. Given a realisation of the environment ξ, we place one particle at x at time $t = 0$. As time evolves, the particle follows the trajectory of a standard Brownian motion $(X_t)_{t\ge 0}$. Additionally and independently of everything else, while at position y, the particle is killed with rate $\xi(y)$. Immediately after its death, the particle is replaced by k independent copies at the site of death, according to some fixed offspring distribution $(p_k)_{k\geq 1}$. All k descendants evolve independently of each other according to the same stochastic diffusion-branching dynamics.

We denote by $\mathtt{P}_x^\xi$ the quenched law of a BBMRE, started at x and write $\mathtt{E}_x^\xi$ for the corresponding expectation. Moreover, we denote by $N(t)$ the set of particles alive at time t. For any particle $\nu \in N(t)$, we denote by $(X_s^\nu)_{s\in[0,t]}$ the spatial trajectory of the genealogy of ancestral particles of ν (unique at any given time) up to time t. Our main interest lies in the maximally displaced particle of the BBMRE at time t,

$$M(t) := \sup\{X_t^\nu : \nu \in N(t)\}.$$

Throughout this article, we deal with supercritical branching such that the offspring distribution has second moments and particles always have at least one offspring.

ASSUMPTION 2. The offspring distribution $(p_k)_{k\geq 1}$ satisfies

$$\sum_{k=1}^{\infty} k p_k =: \mu > 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^2 p_k =: \mu_2 < \infty. \tag{2.5}$$

Under these assumptions, the maximally displaced particle $M(t)$ satisfies a law of large numbers for some nonrandom asymptotic speed $v_0 \in (0, \infty)$, that is, for $\mathbb{P}$-a.e. ξ one has

$$\lim_{t\to\infty} \frac{M(t)}{t} = v_0, \quad \mathtt{P}_0^\xi\text{-a.s.}; \tag{2.6}$$

see [29], Corollary 1.5. (Note also that convergence in probability follows from classical results of Freidlin, [25], Theorem 7.6.1.) The asymptotic speed can be characterised as the unique positive root of the *Lyapunov exponent* λ, which is a deterministic function $\lambda : \mathbb{R} \to \mathbb{R}$ that admits the representation

$$\lambda(v) = \lim_{t\to\infty} \frac{1}{t} \ln \mathtt{E}_0^\xi\big[\big|\{\nu \in N(t) : X_t^\nu \geq vt\}\big|\big], \quad \mathbb{P}\text{-a.s.} \tag{2.7}$$

Under Assumptions 1 and 2, the function λ is nonincreasing on $[0, \infty)$, concave and there exists a critical value $v_c \geq 0$ such that λ is linear on $[0, v_c]$ and strictly concave on $[v_c, \infty)$; see, for example, [20], Corollary 3.10. As in [17, 18, 20], we make the following technical assumption.

ASSUMPTION 3. We only consider BBMREs whose asymptotic speed satisfies

$$v_0 > v_c. \tag{2.8}$$

Essentially, this condition allows for the introduction of a *tilted* probability measure in the *ballistic phase*, under which a Brownian particle $(X_t)_{t\geq 0}$ moves *on average* with speed v_0 up to time t; cf. Section 4. By the same argument as in [17], Lemma A.4, one can show that (2.8) is satisfied by a rich class of environments. We refer also to [20], Section 4.4, for a more in-depth discussion on the condition (2.8) and in particular to [20], Proposition 4.10, where environments are constructed, which satisfy Assumption 1, but for which (2.8) fails. Due to the length of the construction, we do not replicate it here.

Finally, we also define for $\varepsilon \in (0, 1)$ the quenched quantiles for the distribution of $M(t)$ where the process is started at the origin,

$$m_\varepsilon^\xi(t) := \inf\{y \in \mathbb{R} : \mathtt{P}_0^\xi\big(M(t) \leq y\big) \geq \varepsilon\}. \tag{2.9}$$

For notational convenience, we omit the subscript when $\varepsilon = 1/2$ and write $m^\xi(t)$ for the median of the distribution.

With the above notation at our disposal, we can state our main result.

THEOREM 2.1. *Under Assumptions 1–3, for almost every realisation of the environment* ξ, *the family* $(M(t) - m^{\xi}(t))_{t\geq 0}$ *is tight under* P_0^{ξ}.

REMARK 2.2. Note that Theorem 2.1 also remains valid if for any $\varepsilon \in (0, 1)$, the quantity $m^{\xi}(t)$ is replaced by $m_{\varepsilon}^{\xi}(t)$.

This result should be contrasted with the behaviour (1.8) of transition fronts of solutions to the inhomogeneous F-KPP equation (1.7) discussed in the Introduction. More precisely, in [18], Theorems 2.3 and 2.4, environments ξ satisfying Assumptions 1–3 of the present paper were constructed for which the width of the transition front grows logarithmically in time, along a sub-sequence. That is, for small enough $\varepsilon > 0$, there exist times and positions $(t_n)_n, (x_n)_n \in \Theta(n)$, and a function $\varphi \in \Theta(\ln n)$ such that

$$w^{\xi}(t_n, x_n) \geq w^{\xi}\big(t_n, x_n + \varphi(n)\big) + \varepsilon. \tag{2.10}$$

(Note that this not only implies (1.8), but also the spatial nonmonotonicity of the functions $w^{\xi}(t, \cdot)$.) The existence of environments for which Theorem 2.1, and (1.8) or (2.10) hold simultaneously seems unintuitive, as it sharply contrasts with the homogeneous case. Indeed, in the latter, as indicated by (1.4) and (1.5), the standard reasoning for deducing the tightness of BBM is by the uniform boundedness in time of transition fronts for the corresponding homogeneous F-KPP solutions. We will explain the reason for this apparent discrepancy later in the paper (see Section 2.1).

Questions of tightness also arise naturally and have been addressed in many other classes of models. In [15], analytic tools have been developed in order to establish tightness for a class of discrete time models whose distribution functions satisfy certain recursive equations, analogous to the F-KPP equation in the case of BBM. These tools are powerful and were applied and adapted to establish tightness for several models, for example, [1, 9, 19, 23, 30, 44] to name a few. For BBM in a periodic environment, [39] used an analytic result on the F-KPP front in periodic environment [28], which directly implies tightness.

In the context of the discrete space model of [17], sub-sequential tightness along a deterministic sequence is shown for the quenched and annealed law of the maximally displaced particle in [34] using a Dekking–Host type argument. Our method relies crucially on analytic properties of solutions to the F-KPP equation, and differs from the approaches in the above mentioned articles.

The tightness result of Theorem 2.1 naturally suggests the question whether the random variables $M(t) - m^{\xi}(t)$ converge in distribution as $t \to \infty$. Supported by the numerical simulations presented in Figure 1, we conjecture that the answer to this question is negative.

REMARK 2.3. As observed above, in [29] the authors prove an annealed functional central limit theorem for the position of the maximally displaced particle $M(t)$ of BBMRE in the setting described here. In our notation, this means that for some $\sigma^2 \in (0, \infty)$, the sequence of processes

$$[0, \infty) \ni t \mapsto \frac{M(nt) - v_0 nt}{\sqrt{\sigma^2 n}}, \quad n \in \mathbb{N},$$

under $\mathbb{P} \times \mathrm{P}_x^{\xi}$ converges weakly in $C([0, \infty))$ to standard Brownian motion.

Using McKean's representation (see Proposition 3.1 below), reflecting the potential around the origin by defining $\widetilde{\xi}(-y) := \xi(y)$ for all $y \in \mathbb{R}$, and using its stationarity, we obtain that for $x \in \mathbb{R}$ the solution $w^{\widetilde{\xi}}(t, x)$ to (1.7) with initial condition $\mathbf{1}_{(-\infty,0]}$ has the same $\mathbb{P}$-distribution as $\mathrm{P}_0^{\xi}(M(t) \geq x)$. As a consequence, the (functional) central limit theorem [20],

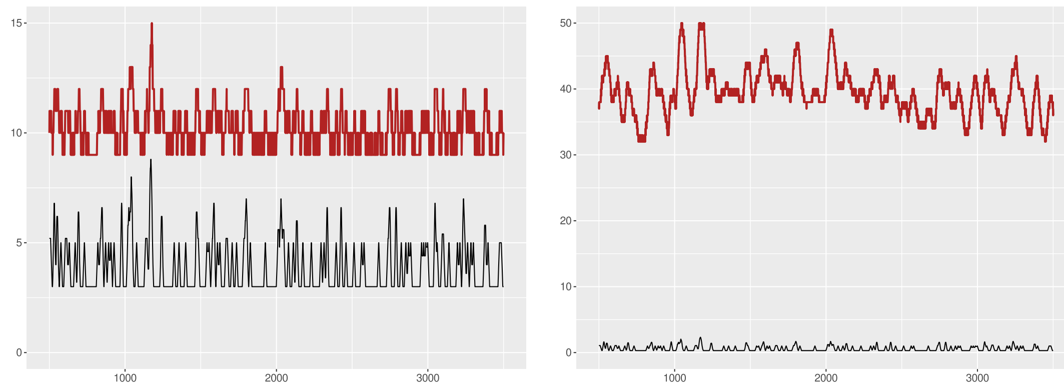

FIG. 1. *Numerical simulations suggesting that the distributions of $M(t) - m^{\xi}(t)$ do not converge as $t \to \infty$; the red line shows the dependence of the "spread" of this distribution, that is, of $m^{\xi}_{0.99}(t) - m^{\xi}_{0.01}(t)$, on the median $m^{\xi}(t)$. The black line shows the corresponding potential $\xi(x)$ as a function of x. The simulations were performed for a discrete-space model, for realisations of ξ from two different distributions (left and right panel). In both cases, note the similarity of the red and black line, in the sense that at times t when the median $m^{\xi}(t)$ reaches an area where ξ is large, the spread of $M(t)$ tends to be large as well.*

Corollary 1.6, for the front of this solution $w^{\widetilde{\xi}}(t,x)$ at level $\varepsilon \in (0,1)$ (note that $\widetilde{\xi}$ still satisfies the conditions imposed on ξ for [20], Corollary 1.6, to hold) entails a (nonfunctional) central limit theorem for $m^{\xi}_{\varepsilon}(t)$ as defined in (2.9) as well, that is, the sequence of random variables $(m^{\xi}(nt) - v_0 nt)/\sqrt{\sigma^2 n}$, $n \in \mathbb{N}$, converges weakly to a $\mathcal{N}(0,\sigma^2)$-distributed random variable. In combination with Theorem 2.1, we recover a nonfunctional form of the above central limit theorem for $M(t)$ as well. That is, the sequence of random variables $(M(nt) - v_0 nt)/\sqrt{\sigma^2 n}$, $n \in \mathbb{N}$, converges weakly to a $\mathcal{N}(0,\sigma^2)$-distributed random variable under $\mathbb{P} \times \mathrm{P}^{\xi}_x$.

2.1. *Strategy of the proof.* We now explain the main ideas behind the proof of Theorem 2.1, and also comment on the seeming discrepancy between this theorem and properties (1.8), (2.10) of the solutions to the randomized F-KPP equation.

The first ingredient of the proof is the well-known duality between the distribution of $M(t)$ and the solutions to the randomised F-KPP equation (1.7). In the spatially nonhomogeneous case this duality states (see Section 3.1 below)

$$w^y(t,x) = \mathrm{P}^{\xi}_x\big(M(t) \ge y\big), \tag{2.11}$$

where w^y is the solution to (1.7) with the initial condition $w^y(0,\cdot) = \mathbf{1}_{[y,\infty)}$. In order to prove the tightness of $(M(t) - m^{\xi}(t))_{t\ge 0}$, it suffices to check that the difference of the ε-quantile and $(1-\varepsilon)$-quantile of $M(t)$ is bounded uniformly in time. More precisely, for every $\varepsilon > 0$ we need to find $\Delta = \Delta(\varepsilon) < \infty$, so that for all $t > 0$ and $x_t = x_t(\varepsilon) \in \mathbb{R}$ characterised via

$$\varepsilon = \mathrm{P}^{\xi}_0\big(M(t) \ge x_t\big) = w^{x_t}(t,0) \tag{2.12}$$

it holds that

$$1 - \varepsilon < \mathrm{P}^{\xi}_0\big(M(t) \ge x_t - \Delta\big) = w^{x_t - \Delta}(t,0). \tag{2.13}$$

We note here that this already provides an indication that the above mentioned discrepancy is only apparent: While properties (1.8) and (2.10) are linked to the dependency of expressions of the form $w^y(t,x)$ on the spatial variable x, the tightness of $M(t)$ is linked to its dependency on the initial condition $\mathbf{1}_{[y,\infty)}$.

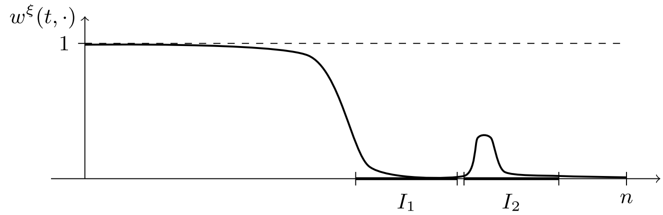

FIG. 2. *A bump in the solution to* (1.7), *which develops shortly after the front of the solution* (*moving to the right*) *reaches the interval* I_1 *where* ξ *is close to* $\texttt{ei}$, *and which is followed by the interval* I_2 *with* ξ *close to* $\texttt{es}$, *if* $\texttt{es}/\texttt{ei} > 2$. *Both intervals are of length* $\Theta(\log n)$.

To show that (2.13) holds true, we first exploit the fact that solutions to (1.7) increase quickly to 1, once they move away from 0. In connection with (2.12), this fact implies (cf. Corollary 3.6 below) that for some $T = T(\varepsilon) < \infty$, uniformly in ξ and t large, we have

$$1 - \varepsilon < w^{x_t}(t + T, 0) =: \overline{w}(t, 0). \tag{2.14}$$

Note that $\overline{w}$ is the solution of (1.7) with the initial condition $w^{x_t}(T, \cdot)$. In view of this, (2.13) follows, if we can show that for some Δ sufficiently large, we have uniformly in t large that

$$w^{x_t - \Delta}(t, 0) > \overline{w}(t, 0). \tag{2.15}$$

Proving inequality (2.15) directly at the spatial coordinate $x = 0$ seems to be difficult. It requires comparing two solutions to the randomised F-KPP equation (1.7) in the regime where they are away from 0 and 1, and where various approximations to them, for example, by linearisation, are not precise enough. To work around this difficulty, we take advantage of the *Sturmian principle* for solutions of parabolic PDEs, which we recall in Section 3.2. As we will see in the proof of Theorem 2.1, this principle implies that (2.15) follows if, for some $v > 0$, we can show the modified inequality

$$w^{x_t - \Delta}(t, -vt) > \overline{w}(t, -vt). \tag{2.16}$$

The advantage of inequality (2.16) is that if v is sufficiently away from 0, then both its sides are very close to 0, and thus can be controlled using linearisation techniques or the first-order Feynman–Kac formulas. The proof of (2.16) is still rather technical and is provided in the Lemma 6.1 below.

Finally, we return to the discrepancy between the divergence of the width of the transition front (1.8) and the tightness proved in Theorem 2.1. The proof of (1.8) in [18] exploits the fact that it is easy to construct potentials where, for n large, in any interval $[0, n]$ there is a subinterval of length $\Theta(\log n)$ where the potential is close to $\texttt{ei}$, closely followed by a subinterval of the same length where the potential almost equals $\texttt{es}$. If $\texttt{es}/\texttt{ei} > 2$, then the existence of such subintervals forces a creation of "bumps" in the solutions to the randomised F-KPP equation, as illustrated in Figure 2. The creation of such bumps directly leads to (1.8). It turns out that the existence of those subintervals does not make inequality (2.16) invalid, even, for example, if x_t is located in a subinterval where the potential is close to $\texttt{es}$ and $x_t - \Delta$ in a subinterval where ξ is almost $\texttt{ei}$. Via the duality (2.11), this is related to the established intuition that the behaviour of the maximum of branching processes is more easily influenced by the environment close to their starting point than by that elsewhere.

Organisation of the article. The rest of the paper is organised as follows. In Section 3, we first recall some well-established facts, such as the duality (2.11), the Feynman–Kac formulas for the solution of the randomised F-KPP equation and of its linearisation, the parabolic Anderson model. We then discuss a first application of the Sturmian principle to our setting.

Section 4 reviews tilted measures, which will later serve, on a technical level, to compare the terms in (2.16). In Section 5, we explain how to extend the arguments in [18, 20] in order to obtain a spatial and temporal perturbation result for solutions of the parabolic Anderson model. This perturbation result is then applied in Section 6 where we prove the key technical lemma, which is related to inequality (2.16). Finally, Section 7 contains the proof of the main theorem.

Notational conventions. We use c, C, c', etc. to denote positive finite constants whose value may change during computations. Indexed versions c_1, c_2, C_1, etc. are used to denote positive finite constants that are fixed throughout the article. Their dependence on other parameters is made explicit when the constants are introduced, with an additional convention given in Section 5.

3. Preliminaries. This section recalls two important and well-known probabilistic tools, which feature heavily in the proof of our main theorem. Furthermore, we make precise the Sturmian principle alluded to above.

3.1. *The randomised F-KPP equation and its linearisation.* As already mentioned in the Introduction, there is a fundamental link between branching Brownian motion and solutions to the homogeneous F-KPP equation. It is often attributed to McKean [42], but can already be found in Skorokhod [46] and Ikeda, Nagasawa and Watanabe [31]. Such a connection can also be extended to the setting of random branching rates, as we now describe. For this purpose, assume we are given an offspring distribution $(p_k)_{k\geq 1}$ as in (2.5). We then consider the random semilinear heat equation

$$\begin{aligned} &\partial_t w(t,x) = \frac{1}{2}\partial_x^2 w(t,x) + \xi(x)F\big(w(t,x)\big), \quad t>0, x\in\mathbb{R}, \\ &w(0,x) = w_0(x), \quad x\in\mathbb{R}, \end{aligned} \tag{F-KPP}$$

where the nonlinearity $F:[0,1]\to[0,1]$ is given by

$$F(w) = (1-w) - \sum_{k=1}^{\infty} p_k(1-w)^k, \quad w\in[0,1]. \tag{3.1}$$

The adaptation of McKean's representation of solutions to (F-KPP) takes the following form.

PROPOSITION 3.1. *For any function $w_0:\mathbb{R}\to[0,1]$, which is the pointwise limit of an increasing sequence of continuous functions, and for any bounded, locally Hölder continuous function $\xi:\mathbb{R}\to(0,\infty)$, there exists a solution to* (F-KPP), *which is continuous on $(0,\infty)\times\mathbb{R}$ and which, for $t\in[0,\infty)$ and $x\in\mathbb{R}$, can be represented as*

$$w(t,x) = 1 - \mathrm{E}_x^{\xi}\Big[\prod_{\nu\in N(t)}\big(1-w_0\big(X_t^{\nu}\big)\big)\Big]. \tag{3.2}$$

A proof of this proposition can be found, for example, in [20], Proposition 2.1; the formulation in that source is under slightly more restrictive conditions, but it transfers verbatim to the assumptions we impose above.

A crucial consequence of Proposition 3.1 is that the solution w^y of (F-KPP) with Heaviside-like initial condition $w_0^y = \mathbf{1}_{[y,\infty)}$, for $y\in\mathbb{R}$, is linked to the distribution function of $M(t)$ via the identity

$$w^y(t,x) = \mathrm{P}_x^{\xi}\big(M(t)\geq y\big). \tag{3.3}$$

REMARK 3.2. It is common practice in the F-KPP literature to normalise the nonlinearity F in such a way that its derivative at the origin is one. Using (2.5), it is easy to check that in our case, $F'(0) = \mu - 1$. In other words, the standard normalisation of equation (F-KPP) corresponds to a branching processes for which the offspring distribution has mean $\mu = 2$, as is also assumed in [20]. In (2.5), we assume only that $\mu > 1$ and do not a priori work under the usual F-KPP normalisation. Nevertheless, given any such offspring distribution $(p_k)_{k\geq 1}$ with mean $\mu \neq 2$ and a corresponding BBMRE in environment ξ, one can always transform it into another BBMRE in a rescaled environment, so that the transformed process is in the usual normalisation and has the same distribution as the original process. Indeed, the transformation defined by

$$\xi \to (\mu - 1)\xi, \qquad p_1 \to \frac{\mu + p_1 - 2}{\mu - 1} \quad \text{and} \quad p_k \to \frac{p_k}{\mu - 1} \quad \text{for } k \geq 2,$$

yields a new offspring distribution with mean two. Moreover, rescaling the environment guarantees that (F-KPP), and the law $\mathtt{P}_x^\xi$ are invariant under this transformation. After rescaling, it holds that $F'(0) = 1$ and $\mu_2 > 2$; hence, in light of this reasoning, we will from now on always assume that

$$\mu = 2, \qquad F'(0) = 1 \quad \text{and} \quad \mu_2 > 2. \tag{3.4}$$

Observe also that by (3.1) this implies that

$$F'(w) \leq 1 \quad \text{and} \quad F''(w) \geq -\mu_2 + 2 \quad \text{for all } w \in [0, 1]. \tag{3.5}$$

Another PDE related to BBMRE, which we make use of later on, is the linearisation of (F-KPP), known as the *parabolic Anderson model* (PAM),

$$\begin{aligned} &\partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) + \xi(x)u(t,x), \quad t > 0, x \in \mathbb{R} \\ &u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{PAM}$$

The PAM has been the subject of intense investigation in its own right (see, e.g., [33] and reference therein for a comprehensive overview); our main interest, however, lies in space and time perturbation results that have been developed for its solution in [18, 20]. These will be considered in more detail in Section 5.

An important strategy for probabilistically investigating the solutions to the equations (F-KPP) and (PAM) is via analysing their Feynman–Kac representations. In what comes below, we denote by P_x the probability measure under which the process denoted by $(X_t)_{t\geq 0}$ is a standard Brownian motion started at $x \in \mathbb{R}$. The corresponding expectation operator is denoted by E_x. We also make repeated use of the abbreviation $E_x[f; A]$ for $E_x[f\mathbf{1}_A]$.

PROPOSITION 3.3. *Under Assumptions 1 and 2, the unique nonnegative solution u of* (PAM) *is given by*

$$u(t,x) = E_x\left[\exp\left\{\int_0^t \xi(X_r)\,\mathrm{d}r\right\}u_0(X_t)\right], \quad t \geq 0, x \in \mathbb{R}, \tag{3.6}$$

and the unique nonnegative solution w of (F-KPP) *fulfils*

$$w(t,x) = E_x\left[\exp\left\{\int_0^t \xi(X_r)\widetilde{F}\big(w(t-r, X_r)\big)\,\mathrm{d}r\right\}w_0(X_t)\right], \quad t \geq 0, x \in \mathbb{R}, \tag{3.7}$$

where $\widetilde{F}(w) = F(w)/w$ for $w \in (0, 1]$, which can be continuously extended to $\widetilde{F}(0) = \lim_{w\to 0+} \widetilde{F}(w) = \sup_{w\in(0,1]} \widetilde{F}(w) = 1$.

See, for example, [13], (1.32), (1.33), for references to the former. Note that the Feynman–Kac representation (3.6) is explicit, while (3.7) is not (in the sense that the expressions on both sides of the latter equation involve w).

Taking advantage of the above, a (direct) link between the PAM and BBMRE can be derived by combining the Feynman–Kac representation (3.6) of the solution to (PAM) with a *many-to-one formula* (see, e.g., [20], Proposition 2.3), in order to arrive at the representation

$$u(t,x) = \mathrm{E}_x^{\xi}\Big[\sum_{\nu \in N(t)} u_0\big(X_t^{\nu}\big)\Big]$$

of solutions to (PAM).

3.2. *Sturmian principle.* In this section, we present the analytic ingredient of our proof of Theorem 2.1. As explained in the Introduction (see around (2.15)), we are interested in differences of the type $W(\cdot,\cdot) = w^{y_1}(\cdot,\cdot) - w^{y_2}(\cdot + T, \cdot)$ for some $T > 0$, and $y_2 > y_1$, where we recall that for any $y \in \mathbb{R}$, we denote by w^y the solution of (F-KPP) with initial condition $w_0 = \mathbf{1}_{[y,\infty)}$. It is immediate that the function W satisfies the linear parabolic equation

$$\begin{aligned} \partial_t W(t,x) &= \frac{1}{2}\partial_x^2 W(t,x) + G(t,x)W(t,x), \quad t > 0, x \in \mathbb{R}, \\ W(0,x) &= \mathbf{1}_{[y_1,\infty)}(x) - w^{y_2}(T,x), \quad x \in \mathbb{R}, \end{aligned} \tag{3.8}$$

where G is the bounded measurable function defined by (using the convention $F'(0) = 1$, cf. Remark 3.2)

$$G(t,x) = \begin{cases} \xi(x)\dfrac{F(w^{y_1}(t,x)) - F(w^{y_2}(t+T,x))}{w^{y_1}(t,x) - w^{y_2}(t+T,x)} & \text{if } w^{y_1}(t,x) \neq w^{y_2}(t+T,x), \\ \xi(x) & \text{if } w^{y_1}(t,x) = w^{y_2}(t+T,x). \end{cases} \tag{3.9}$$

Let us state the following simple observation, which will be used at various stages in the following: By Proposition 3.1, it follows that

$$0 < w^{y_2}(T,x) < 1 \quad \text{for all } T > 0 \text{ and } x \in \mathbb{R}. \tag{3.10}$$

As a consequence, the initial condition of (3.8) has exactly one zero-crossing, and it is located at y_1.

In the analysis literature, it has been known for a long time that the cardinality of the set of zero-crossings of solutions to linear parabolic equations is monotonically nonincreasing in time, with the earliest reference dating back to at least an article by Charles Sturm in 1836; cf. [47]. Nevertheless, despite this result having been known for almost two centuries by now, it was not until the eighties of the last century that Sturm's ideas really revived in the theory of linear and nonlinear parabolic equations; see,for example, [4, 5, 21, 22, 43] for a nonexhaustive list. In this list, the ideas in [22] stand out, as they involve a simple and purely probabilistic proof by interpreting the linear parabolic partial differential equations as generators of Markov processes and reducing the study of the zero-crossings to the study of Markovian transition operators acting on signed measure spaces. A more complete history and a detailed discussion of the Sturmian principle and its applications can be found in [26]. In this context, it is interesting to note that already in their seminal article on the F-KPP equation, Kolmogorov, Petrovskii and Piskunov also make use of a Sturmian principle for equations of the form (3.8) (see [32], Theorem 11), which is proved using a parabolic maximum principle.

We include a version of such a result, which is formulated to fit our purpose; a more general version of this result can be found in [43]. Note that the assumptions in particular fit the setting of a single zero-crossing in the initial datum.

LEMMA 3.4 ([43], Proposition 7.1). *For any $t_0 \in \mathbb{R}$, let $G \in L^\infty((t_0,\infty)\times\mathbb{R})$ and assume $W \in C((t_0,\infty)\times\mathbb{R}) \cap L^\infty((t_0,\infty)\times\mathbb{R})$ to be a weak solution of*

$$\partial_t W(t,x) = \frac{1}{2}\partial_x^2 W(t,x) + G(t,x)W(t,x), \quad t > t_0, x \in \mathbb{R},$$
$$W(t_0,x) = W_{t_0}(x), \quad x \in \mathbb{R},$$

where $W_{t_0} \not\equiv 0$ is piecewise continuous and bounded in $\mathbb{R}$, such that for some $z_{t_0} \in \mathbb{R}$ one has

$$W_{t_0}(x) \le 0 \quad \text{if } x < z_{t_0} \quad \text{and} \quad W_{t_0}(x) \ge 0 \quad \text{if } x > z_{t_0}.$$

Then, for all $t > t_0$ there exists a unique point $z(t) \in [-\infty,\infty]$ such that

$$W(t,x) < 0 \quad \text{if } x < z(t) \quad \text{and} \quad W(t,x) > 0 \quad \text{if } x > z(t).$$

As a first application of Lemma 3.4, let us consider the behaviour of the solutions of (F-KPP) when the discontinuity of the Heaviside-type initial condition tends to infinity. For this purpose, in order to obtain a nontrivial limit, we perform an appropriate temporal shift. More precisely, we introduce for a given realisation of the environment ξ, any $y \in \mathbb{R}$ and any $\varepsilon > 0$ the "temporal quantile at the origin" as

$$\tau_y^\varepsilon := \inf\{t \ge 0 : w^y(t,0) \ge \varepsilon\}. \tag{3.11}$$

Since $\mathbb{P}$-a.s., we have $\lim_{t\to\infty} w^y(t,0) = 1$ (due to, e.g., [25], Theorem 7.6.1), τ_y^ε is finite. By the continuity of w^y on $(0,\infty)\times\mathbb{R}$ (cf. Proposition 3.1), the quantity τ_y^ε satisfies $w^y(\tau_y^\varepsilon, 0) = \varepsilon$. From (3.3), it follows that $y \mapsto w^y(t,0)$ is decreasing, and thus $y \mapsto \tau_y^\varepsilon$ is increasing. By the law of large numbers for the maximal displacement (cf. (2.6)), it follows readily that $\lim_{y\to\infty} \tau_y^\varepsilon = \infty$.

The shift by τ_y^ε allows to establish the following result, which follows already from [43], Lemma 7.3. Nevertheless, we provide its short proof here for the sake of completeness and as an illustration of how Lemma 3.4 can be used in our context.

PROPOSITION 3.5. *For every $\varepsilon \in (0,1)$, $\mathbb{P}$-a.s., the limit*

$$w_\varepsilon^\infty(t,x) := \lim_{y\to\infty} w^y(\tau_y^\varepsilon + t, x) \tag{3.12}$$

exists locally uniformly in $(t,x) \in \mathbb{R}^2$, and is a global-in-time (i.e., for all $t \in \mathbb{R}$) solution to (F-KPP).

The limiting function w_ε^∞ plays a role comparable to that of a travelling wave solution of the homogeneous F-KPP equation; cf. (1.3). However, unlike in the homogeneous situation outlined in the Introduction, w_ε^∞ does not directly provide an argument for tightness because we lack a suitable quantitative control of the random variables τ_y^ε as y varies. Nonetheless, the result of Proposition 3.5 plays a vital role in our proof of tightness. We restrict ourselves to providing a proof of the convergence for $t > 0$ only, as this is sufficient for our purposes in what follows.

PROOF OF PROPOSITION 3.5. Fix $y_1 < y_2$ and for $t \ge -\tau_{y_1}^\varepsilon = -\tau_{y_1}^\varepsilon \vee -\tau_{y_2}^\varepsilon$ (recall that the latter identity follows from the monotonicity of $y \mapsto \tau_y^\varepsilon$ observed below (3.11)) define the function $W(t,x) := w^{y_1}(t+\tau_{y_1}^\varepsilon, x) - w^{y_2}(t+\tau_{y_2}^\varepsilon, x)$. Then, similarly as for (3.8) and (3.9), it follows that

$$\partial_t W(t,x) = \frac{1}{2}\partial_x^2 W(t,x) + G(t,x)W(t,x), \quad t > -\tau_{y_1}^\varepsilon, x \in \mathbb{R}, \tag{3.13}$$

where G is given by

$$G(t,x) = \begin{cases} \xi(x)\dfrac{F(w^{y_1}(t+\tau^{\varepsilon}_{y_1},x)) - F(w^{y_2}(t+\tau^{\varepsilon}_{y_2},x))}{w^{y_1}(t+\tau^{\varepsilon}_{y_1},x) - w^{y_2}(t+\tau^{\varepsilon}_{y_2},x)} \\ \qquad \text{if } w^{y_1}(t+\tau^{\varepsilon}_{y_1},x) \neq w^{y_2}(t+\tau^{\varepsilon}_{y_2},x), \\ \xi(x) \quad \text{if } w^{y_1}(t+\tau^{\varepsilon}_{y_1},x) = w^{y_2}(t+\tau^{\varepsilon}_{y_2},x). \end{cases}$$

From the assumptions, it follows directly that G is a bounded measurable function. Due to [25], Theorem 7.4.1, there exists for $\mathbb{P}$-a.e. ξ a unique classical solution to (3.13). Moreover, since $w^{y_1}(0,x) = \mathbf{1}_{[y_1,\infty)}(x)$, it holds that

$$W(-\tau^{\varepsilon}_{y_1},x) = w^{y_1}(0,x) - w^{y_2}(\tau^{\varepsilon}_{y_2} - \tau^{\varepsilon}_{y_1},x) = \mathbf{1}_{[y_1,\infty)}(x) - w^{y_2}(\tau^{\varepsilon}_{y_2} - \tau^{\varepsilon}_{y_1},x). \tag{3.14}$$

Together with the fact that $0 < w^{y_i}(t,x) < 1$ for $i = 1,2$ and for all $t > 0$ and $x \in \mathbb{R}$ (cf. (3.10)), (3.14) implies that $W(-\tau^{\varepsilon}_{y_1},x) < 0$ if $x < y_1$ and $W(-\tau^{\varepsilon}_{y_1},x) > 0$ if $x > y_1$. By Lemma 3.4, for all $t > -\tau^{\varepsilon}_{y_1}$, the sets $\{x \in \mathbb{R} : W(t,x) > 0\}$ and $\{x \in \mathbb{R} : W(t,x) < 0\}$ are intervals. But due to the continuity of w^{y_1} and w^{y_2}, we also know that $W(0,0) = w^{y_1}(\tau^{\varepsilon}_{y_1},0) - w^{y_2}(\tau^{\varepsilon}_{y_2},0) = \varepsilon - \varepsilon = 0$. Therefore, the above reasoning supplies us with

$$\begin{aligned} w^{y_1}(\tau^{\varepsilon}_{y_1},x) &\le w^{y_2}(\tau^{\varepsilon}_{y_2},x) \quad \text{if } x < 0, \\ w^{y_1}(\tau^{\varepsilon}_{y_1},x) &\ge w^{y_2}(\tau^{\varepsilon}_{y_2},x) \quad \text{if } x > 0. \end{aligned} \tag{3.15}$$

That is, the function $y \mapsto w^y(\tau^{\varepsilon}_y,x)$ is nondecreasing if $x < 0$ and nonincreasing on $x > 0$. As a consequence, the limit $w^{\infty}_{\varepsilon}(0,x) := \lim_{y\to\infty} w^y(\tau^{\varepsilon}_y,x)$ exists pointwise, and thus locally uniformly, for all $x \in \mathbb{R}$, and also implies $0 \le w^{\infty}_{\varepsilon}(0,\cdot) \le 1$. As a consequence, the right-hand side of (3.12) converges locally uniformly for $t = 0$. (This should be compared to (1.4) in the Introduction, which describes the "spatial stretching" of recentred solutions to the homogeneous F-KPP equation.)

To prove that the local uniform convergence postulated in (3.12) holds true for $t > 0$ also, one uses standard estimates on solutions of quasilinear parabolic equations (see, e.g., [36], Chapter V). As a consequence of these estimates, the solutions $w^y(t,x)$ together with their derivatives are bounded locally uniformly in (t,x), uniformly for all y sufficiently large. Hence, the set $\{w^y : y \ge 0\}$ is precompact in $C^{1,2}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$. It therefore contains converging sub-sequences, and every limit point of such a sub-sequence is a solution to (F-KPP) with initial condition $w^{\infty}(0,\cdot)$. As the solution to (F-KPP) with that given initial condition is unique, this implies that all subsequential limits must agree, and thus (3.12) holds for all $t > 0$, as well as that w^{∞} solves (F-KPP) for $t \ge 0$. We omit here the proof for $t < 0$, as it will not be needed later on. □

The next corollary is a direct consequence of Proposition 3.5. It formalises the idea that when a solution to (F-KPP) moves away from 0, it increases quickly to 1. This is going to be relevant later on (cf. (2.14) in the Introduction).

COROLLARY 3.6. *For every $\varepsilon \in (0,1/2)$, there exists a $\mathbb{P}$-a.s. finite random variable $T = T(\xi)$ such that for all $y \in \mathbb{R}$ large enough, and any t for which $w^y(t,0) = \varepsilon$, it holds that*

$$w^y(t+t',0) \ge 1 - \varepsilon \quad \textit{for all } t' \in [T, T+1].$$

PROOF. Let $y \in \mathbb{R}$ and $t \ge 0$ be such that $w^y(t,0) = \varepsilon$. By (3.11) and the finiteness of τ^{ε}_y deduced below that display, there exists some $s_0 = s_0(y) \ge 0$ such that $t = \tau^{\varepsilon}_y + s_0$.

Consider w_ε^∞ from Proposition 3.5 and let

$$s_1 = \inf\{s > s_0 : w_\varepsilon^\infty(s', 0) \geq 1 - \varepsilon/2 \text{ for all } s' > s\};$$

note that as w_ε^∞ solves (F-KPP), it follows by [25], Theorem 7.6.1, that for $\mathbb{P}$-a.e. realisation of the environment, $\lim_{s\to\infty} w_\varepsilon^\infty(s, x) = 1$, and hence s_1 is $\mathbb{P}$-a.s. finite. Next, taking advantage of the fact that the convergence in Proposition 3.5 is locally uniform in t, due to the continuity of the functions involved and using the compactness of $[s_1, s_1 + 1]$, it holds for large enough $y \in \mathbb{R}$ that

$$\sup_{s' \in [s_1, s_1+1]} \left| w^y(\tau_y^\varepsilon + s', 0) - w_\varepsilon^\infty(s', 0) \right| < \varepsilon/2.$$

Setting $T = s_1 - s_0$, we thus obtain for all y large enough and for all $t' \in [T, T + 1]$ (with $s' = s_0 + t' \in [s_1, s_1 + 1]$) that

$$w^y(t + t', 0) = w^y(\tau_y^\varepsilon + s', 0) \geq w_\varepsilon^\infty(s', 0) - \varepsilon/2 \geq 1 - \varepsilon.$$

This completes the proof. □

This result concludes our analytic preparations on how the set of zero-crossings of solutions to linear parabolic equations evolves, and of how it can be applied to the difference of temporally shifted solutions of (F-KPP).

4. Tilting and exponential change of measure. The next tool that we introduce is a change of measure for Brownian paths in the Feynman–Kac representation, which makes certain large deviation events typical. These measures have been featured heavily in [17, 18, 20] already, including in the proof of their respective versions of Proposition 5.1. In the aforementioned articles, this change of measure has been employed so as to make solutions to (PAM) amenable to the investigation by more standard probabilistic tools. Here, we go a step further and consider the stochastic processes driving the tilted path measures. This, in turn, gives us even more precise control on the tilted measures and allows for comparisons with Brownian motion with constant drift; see Proposition 4.2 below.

To define the tilted measures, we set

$$\zeta := \xi - \mathtt{es}. \tag{4.1}$$

Due to the uniform boundedness (2.4), it follows that $\mathbb{P}$-a.s. for all $x \in \mathbb{R}$,

$$\zeta(x) \in [\mathtt{ei} - \mathtt{es}, 0], \tag{4.2}$$

and ζ is $\mathbb{P}$-a.s. locally Hölder continuous with the same exponent as ξ. Moreover, ζ also inherits the stationarity as well as the mixing property from ξ.

For the Brownian motion $(X_t)_{t\geq 0}$ under the measure P_x, as used in the Feynman–Kac representations of Proposition 3.3, we introduce first hitting times as

$$H_y := \inf\{t \geq 0 : X_t = y\} \quad \text{for } y \in \mathbb{R}. \tag{4.3}$$

Analogous to [17, 18, 20], we define for $x \leq y \in \mathbb{R}$ and $\eta < 0$ the *tilted* path measures characterised through events $A \in \sigma(X_{t\wedge H_y}, t \geq 0)$ via

$$P_{x,y}^{\zeta,\eta}(A) := \frac{1}{Z_{x,y}^{\zeta,\eta}} E_x\big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}; A\big], \tag{4.4}$$

with normalising constant

$$Z_{x,y}^{\zeta,\eta} := E_x\big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\big] \in (0, 1]. \tag{4.5}$$

By the strong Markov property, it follows easily that the measures are consistent in the sense that $P^{\zeta,\eta}_{x,y'}(A) = P^{\zeta,\eta}_{x,y}(A)$ for $x \le y \le y'$ and $A \in \sigma(X_{t\wedge H_y}, t \ge 0)$. Hence, for any $x \in \mathbb{R}$, we can extend $P^{\zeta,\eta}_{x,y}$ to a

$$\text{probability measure } P^{\zeta,\eta}_x \text{ on } \sigma(X_t, t \ge 0) \tag{4.6}$$

with the help of Kolmogorov's extension theorem. We write $E^{\zeta,\eta}_x$ for the expectation with respect to the probability measure $P^{\zeta,\eta}_x$.

Finally, as in [20], (2.8), we introduce the annealed logarithmic moment generating function

$$L(\eta) := \mathbb{E}[\ln Z^{\zeta,\eta}_{0,1}], \tag{4.7}$$

and for denote by $\overline{\eta}(v) < 0$ the unique solution of the equation $L'(\overline{\eta}(v)) = \frac{1}{v}$ for any $v > v_c$ (where v_c is as in Assumption 3). Observe that $\overline{\eta}$ is well-defined as by [20], Lemma 2.4,

(4.8) $\overline{\eta}(v)$ exists for every $v > v_c$; $v \mapsto \overline{\eta}(v)$ is a continuous decreasing function and $\lim_{v\to\infty} \overline{\eta}(v) = -\infty$.

The strong Markov property furthermore entails that, for a fixed realisation ζ and any $\eta < 0$, the normalising constants (4.5) are multiplicative in the sense that for any $x < y < z$ in $\mathbb{R}$,

$$Z^{\zeta,\eta}_{x,z} = Z^{\zeta,\eta}_{x,y} Z^{\zeta,\eta}_{y,z}. \tag{4.9}$$

Defining, for some arbitrary but fixed $x_0 \in \mathbb{R}$, the function

$$Z^{\zeta,\eta}(x) := \begin{cases} (Z^{\zeta,\eta}_{x_0,x})^{-1} & \text{if } x \ge x_0, \\ Z^{\zeta,\eta}_{x,x_0} & \text{if } x < x_0, \end{cases} \tag{4.10}$$

the identity (4.9) thus implies that for all $x < y$ we have

$$Z^{\zeta,\eta}_{x,y} = \frac{Z^{\zeta,\eta}(x)}{Z^{\zeta,\eta}(y)}. \tag{4.11}$$

The following lemma states some useful properties of the function $Z^{\zeta,\eta}$.

LEMMA 4.1. *For every locally Hölder continuous function* $\zeta : \mathbb{R} \to [-(\mathtt{es}-\mathtt{ei}), 0]$ *and* $\eta < 0$, *the function* $Z^{\zeta,\eta}$ *is nondecreasing*, *strictly positive*, *twice continuously differentiable and satisfies*

$$\frac{1}{2}\Delta Z^{\zeta,\eta}(x) + \big(\zeta(x) + \eta\big) Z^{\zeta,\eta}(x) = 0, \quad x \in \mathbb{R}. \tag{4.12}$$

Furthermore,

$$b^{\zeta,\eta}(x) := \frac{\mathrm{d}}{\mathrm{d}x} \ln Z^{\zeta,\eta}(x) \in \big[\underline{v}(\eta), \overline{v}(\eta)\big], \tag{4.13}$$

where $\underline{v}(\eta) := \sqrt{2|\eta|}$ *and* $\overline{v}(\eta) := \sqrt{2(\mathtt{es} - \mathtt{ei} + |\eta|)}$.

PROOF. The monotonicity and the strict positivity of $Z^{\zeta,\eta}$ follow directly from its definition (4.10), using also (4.5).

To show (4.12), we observe that, for any interval $[x_1, x_2]$, the equation $\frac{1}{2}\Delta u(x) + (\zeta(x) + \eta)u(x) = 0$, $x \in [x_1, x_2]$, with boundary conditions $u(x_i) = Z^{\zeta,\eta}(x_i)$, $i = 1, 2$, has a unique classical solution (see, e.g., [27], Corollary 6.9). Denoting by T the exit time of X from $[x_1, x_2]$, this solution can be represented as (see [6], Theorem II(4.1), p. 48)

$$u(x) = E_x\big[Z^{\zeta,\eta}(X_T)e^{\int_0^T(\zeta(X_s)+\eta)\,\mathrm{d}s}\big]. \tag{4.14}$$

On the other hand, for $x \in [x_1, x_2]$, taking $y = x_2$ in (4.11), using (4.5), and the strong Markov property at time T,

$$\begin{aligned} Z^{\zeta,\eta}(x) &= Z^{\zeta,\eta}(y)Z^{\zeta,\eta}_{x,y} \\ &= Z^{\zeta,\eta}(y)E_x\big[e^{\int_0^T(\zeta(X_s)+\eta)\,\mathrm{d}s}Z^{\zeta,\eta}_{X_T,y}\big] \\ &= E_x\big[Z^{\zeta,\eta}(X_T)e^{\int_0^T(\zeta(X_s)+\eta)\,\mathrm{d}s}\big]. \end{aligned} \tag{4.15}$$

Therefore, $Z^{\zeta,\eta}$ satisfies (4.12) on $[x_1, x_2]$. Since the interval $[x_1, x_2]$ is arbitrary, (4.12) holds for every $x \in \mathbb{R}$.

To show (4.13), note first that $b^{\zeta,\eta}$ is well-defined since $Z^{\zeta,\eta}$ is strictly positive and differentiable, by (4.12). Therefore, with $y \geq x$, by (4.11) and the strong Markov property again,

$$\begin{aligned} b^{\zeta,\eta}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\ln Z^{\zeta,\eta}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln Z^{\zeta,\eta}_{x,y} \\ &= \lim_{\varepsilon\to0^+}\varepsilon^{-1}\big(\ln E_x\big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\big] - \ln E_{x-\varepsilon}\big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\big]\big) \\ &= -\lim_{\varepsilon\to0^+}\varepsilon^{-1}\ln E_{x-\varepsilon}\big[e^{\int_0^{H_x}(\zeta(X_s)+\eta)\,\mathrm{d}s}\big]. \end{aligned} \tag{4.16}$$

It is a known fact that for $\alpha > 0$ and $z_1, z_2 \in \mathbb{R}$, it holds that

$$\ln E_{z_1}\big[e^{-\alpha H_{z_2}}\big] = -\sqrt{2\alpha}|z_1 - z_2| \tag{4.17}$$

(cf. [10], (2.0.1), p. 204). In combination with the bounds (4.2), the expectation on the right-hand side of (4.16) thus satisfies

$$\begin{aligned} -\varepsilon\sqrt{2|\eta|} = \ln E_{x-\varepsilon}\big[e^{H_x\eta}\big] &\geq \ln E_{x-\varepsilon}\big[e^{\int_0^{H_x}(\zeta(X_s)+\eta)\,\mathrm{d}s}\big] \\ &\geq \ln E_{x-\varepsilon}\big[e^{H_x(\mathtt{ei}-\mathtt{es}+\eta)}\big] = -\varepsilon\sqrt{2(\mathtt{es}-\mathtt{ei}+|\eta|)}, \end{aligned} \tag{4.18}$$

which together with (4.16) implies (4.13). □

The function $b^{\zeta,\eta}(x)$ introduced in (4.13) is useful in describing the law of X under the tilted measure, as it allows an interpretation of the tilted process as a Brownian motion with an inhomogeneous drift, by constructing an appropriate SDE as follows.

PROPOSITION 4.2. *Let $x_0 \in \mathbb{R}$, $\eta < 0$ and let $\zeta : \mathbb{R} \to (-\infty, 0]$ be a locally Hölder continuous function that is uniformly bounded from below. Furthermore, denote by B a standard Brownian motion. Then the distribution of the solution to the SDE*

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}B_t + b^{\zeta,\eta}(X_t)\,\mathrm{d}t, \quad t > 0, \\ X_0 &= x_0, \end{aligned} \tag{4.19}$$

agrees with $P^{\zeta,\eta}_{x_0}$.

PROOF. The proof is based on an exponential change of measure for diffusion processes. For the sake of simplicity, we write b for $b^{\zeta,\eta}$ and Z for $Z^{\zeta,\eta}$ whenever there is no risk of confusion. By (4.13) we obtain that

$$b' = (\ln Z)'' = \left(\frac{Z'}{Z}\right)' = \frac{\Delta Z}{Z} - \left(\frac{Z'}{Z}\right)^2 = -2(\zeta+\eta) - b^2. \tag{4.20}$$

Therefore, the bounds (4.13) and (4.2) imply that b is a bounded Lipschitz function and thus there is a strong solution to (4.19), whose distribution we denote by $Q_{x_0} = Q_{x_0}^{\zeta,\eta}$. Let further, as previously, P_{x_0} be the distribution of Brownian motion started from x_0, and let $Q_{x_0}^t$ and $P_{x_0}^t$ be the restrictions of those distributions to the time interval $[0,t]$, $t>0$. As a consequence of the Cameron–Martin–Girsanov theorem (see, e.g., [45], Theorem V.27.1, for a suitable formulation), it is well known that

$$\frac{\mathrm{d}Q_{x_0}^t}{\mathrm{d}P_{x_0}^t} = \exp\left\{\int_0^t b(X_s)\,\mathrm{d}X_s - \frac{1}{2}\int_0^t b^2(X_s)\,\mathrm{d}s\right\} =: M_t, \tag{4.21}$$

for a P_{x_0}-martingale M. (The fact that M_t is a martingale follows, e.g., from [45], Theorem IV.37.8, since b is a bounded function.)

With the aim of arriving at a comparison with (4.4), we claim that

$$M_t = \frac{Z(X_t)}{Z(X_0)} e^{\int_0^t (\zeta(X_s)+\eta)\,\mathrm{d}s}. \tag{4.22}$$

To see this, note first that applying Itô's formula to $\ln Z(x) = \int_{x_0}^x b(t)\,\mathrm{d}t$ yields

$$\frac{Z(X_t)}{Z(X_0)} = \exp\{\ln Z(X_t) - \ln Z(X_0)\} = \exp\left\{\int_0^t b(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t b'(X_s)\,\mathrm{d}s\right\}. \tag{4.23}$$

Comparing this with (4.21) shows that

$$M_t = \frac{Z(X_t)}{Z(X_0)} \exp\left\{-\frac{1}{2}\int_0^t \bigl(b'(X_s) + b^2(X_s)\bigr)\,\mathrm{d}s\right\}, \tag{4.24}$$

which together with (4.20) implies (4.22).

We can now complete the proof of the proposition. For the sake of clarity, we sometimes write expectations with respect to a probability measure Q as E^Q. For $y \ge x_0$, let $Q_{x_0,y}$ be the measure Q_{x_0} restricted to the σ-algebra $\mathcal{H}_y = \sigma(X_{s\wedge H_y} : s \ge 0)$. To show that $Q_{x_0} = P_{x_0}^{\zeta,\eta}$, it is sufficient to show that $Q_{x_0,y} = P_{x_0,y}^{\zeta,\eta}$ for all $y > x_0$ (see (4.4)). For this purpose, we observe that by Lemma 4.1, Z is a bounded function on $(-\infty, y]$, and thus the stopped martingale $M_t^{H_y} = M_{t\wedge H_y}$ is uniformly bounded from above. Therefore, by the optional stopping theorem, for any $A \in \mathcal{H}_y$, using (4.21) for the second equality,

$$\begin{aligned}
Q_{x_0,y}(A) &= \lim_{t\to\infty} E^{Q_{x_0,y}}[\mathbf{1}_{A\cap\{H_y\le t\}}] = \lim_{t\to\infty} E^{P_{x_0}}[M_t \mathbf{1}_{A\cap\{H_y\le t\}}] \\
&= \lim_{t\to\infty} E^{P_{x_0}}\bigl[E^{P_{x_0}}[M_t \mathbf{1}_{A\cap\{H_y\le t\}} | \mathcal{H}_y]\bigr] \\
&= \lim_{t\to\infty} E^{P_{x_0}}\bigl[\mathbf{1}_{A\cap\{H_y\le t\}} E^{P_{x_0}}[M_t | \mathcal{H}_y]\bigr] \\
&= \lim_{t\to\infty} E^{P_{x_0}}[\mathbf{1}_{A\cap\{H_y\le t\}} M_{H_y}] = E^{P_{x_0}}[M_{H_y}\mathbf{1}_A].
\end{aligned}$$

By (4.22), $M_{H_y} = \frac{Z(y)}{Z(x_0)} e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}$, and thus, also by (4.11),

$$Q_{x_0,y}(A) = \bigl(Z_{x_0,y}^{\zeta,\eta}\bigr)^{-1} E_{x_0}\bigl[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\mathbf{1}_A\bigr] = P_{x_0,y}^{\zeta,\eta}(A)$$

as required. This completes the proof. □

Proposition 4.2 together with the uniform bounds (4.13) on $b^{\zeta,\eta}$ allows for a comparison between the tilted measures (4.4) and Brownian motion with constant drift. The next lemma makes this precise. For a given drift $\alpha \in \mathbb{R}$, we write P_x^α for the law of Brownian motion with constant drift α started at x and E_x^α for the corresponding expectation.

LEMMA 4.3. *Let $\zeta : \mathbb{R} \to [-(\mathtt{es} - \mathtt{ei}), 0]$ be locally Hölder continuous and let $\eta < 0$. Then, for any starting point $x \in \mathbb{R}$ and any bounded nondecreasing function $g : \mathbb{R} \to \mathbb{R}$,*

$$E_x^{\underline{v}(\eta)}[g(X_t)] \le E_x^{\zeta,\eta}[g(X_t)] \le E_x^{\overline{v}(\eta)}[g(X_t)],$$

where $\underline{v}(\eta)$ and $\overline{v}(\eta)$ have been introduced in Lemma 4.1.

PROOF. By Proposition 4.2, the process X_t driven by the tilted measure $P_{x_0}^{\zeta,\eta}$ has generator $L^{\zeta,\eta} = \frac{1}{2}\Delta + b(x)\frac{\mathrm{d}}{\mathrm{d}x}$. Let further $L^v = \frac{1}{2}\Delta + v\frac{\mathrm{d}}{\mathrm{d}x}$ be the generator of the Brownian motion with drift v. Then, for any nondecreasing $g \in C_b^2(\mathbb{R})$, it follows from (4.13) that

$$L^{\underline{v}(\eta)} g \le L^{\zeta,\eta} g \le L^{\overline{v}(\eta)} g.$$

Since, by Kolmogorov's forward equation, $\frac{\mathrm{d}}{\mathrm{d}t} E_x^{\zeta,\eta}[g(X_t)] = E_x^{\zeta,\eta}[(L^{\zeta,\eta} g)(X_t)]$ and analogously for the measures $E_x^{\underline{v}}$ and $E_x^{\overline{v}}$, the statement of the lemma follows for any nondecreasing $g \in C_b^2(\mathbb{R})$. The extension to arbitrary bounded nondecreasing functions g follows by approximating g by a sequence of nondecreasing functions in $C_b^2(\mathbb{R})$ and using the dominated convergence theorem. □

5. Perturbations of the Feynman–Kac representation. In this section, we provide a regularity type result for the Feynman–Kac representation (3.6) of solutions to the parabolic Anderson model (PAM) with initial conditions of Heaviside type. A variant of such results was developed in [18, 20] (cf. Lemmas 3.11 and 3.13 from [20], or Lemma 4.1 of [18]) for the study of the fronts of (F-KPP) and (PAM). In the current setting, the perturbation results will be used together with the identity (3.3) and the Feynman–Kac representation (Proposition 3.3) in order to get bounds on the solutions to (F-KPP) in the proof of the fundamental Lemma 6.1 in Section 6.

To avoid the dependence of various constants appearing in these perturbation results on the speed, we assume for the rest of the article that the speeds we allow are contained in some arbitrary but fixed compact interval $V \subset (v_c, \infty)$, which has v_0 in its interior (recall that we require (2.8) to hold). As we can otherwise choose V arbitrarily large, this does not impose any further restrictions for what follows. With $\overline{\eta}(v) < 0$ as in (4.8), we further fix a compact interval $\triangle \subset (-\infty, 0)$ such that

(5.1) $$\text{the set } \{\overline{\eta}(v) : v \in V\} \text{ is contained in the interior of } \triangle.$$

From now on, all newly introduced constants may depend implicitly on the compact sets V and $\triangle$ and on the bounds on the environment $\mathtt{es}$ and $\mathtt{ei}$. Their dependence on other parameters is made explicit when the constants are introduced. In order to facilitate reading, for quantities that depend on the realisation of the environment ξ we use letters $\mathcal{T}$, $\mathcal{N}$, etc.

PROPOSITION 5.1.

(a) *For every $\delta > 0$ and $A > 0$, there exist a constant $C_1 = C_1(A, \delta) \in (1, \infty)$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_1 = \mathcal{T}_1(A, \delta)$ such that for all $t \ge \mathcal{T}_1$, uniformly in $0 \le h \le t^{1-\delta}$ and $x, y \in [-At, At]$ with $x < y$, $\frac{y-x}{t} \in V$ and $\frac{y-x}{t+h} \in V$,*

$$E_x\big[e^{\int_0^{t+h} \xi(X_s)\,\mathrm{d}s}; X_{t+h} \ge y\big] \le C_1 e^{C_1 h} E_x\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \ge y\big].$$

(b) *Let $\delta : (0,\infty) \to (0,\infty)$ be a function tending to* 0 *as $t \to \infty$, and let $A > 0$. Then there exists a constant $C_2 = C_2(A,\delta) \in (1,\infty)$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_2 = \mathcal{T}_2(A,\delta)$ such that for all $t \geq \mathcal{T}_2$, uniformly in $0 \leq h \leq t\delta(t)$ and $x, y \in [-At, At]$ with $x < y$, $\frac{y-x}{t} \in V$ and $\frac{y+h-x}{t} \in V$,*

$$E_x\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y+h\big] \leq C_2 e^{-h/C_2} E_x\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y\big].$$

The proof of this proposition involves comparing the Feynman–Kac representation (3.6) to functionals with respect to the family of tilted probability measures that were presented in Section 4. It is a rather straightforward, but lengthy adaptation of parts of the proofs of Lemma 3.11(b) of [20] (which corresponds to part (a) of Proposition 5.1) and Lemma 4.1(b) of [18] (corresponding to part (b) of Proposition 5.1). There are two key differences in the statements of our Proposition 5.1 when compared to the two respective statements in [18, 20], that need to be addressed:

(A) Proposition 5.1 requires that its estimates hold uniformly over the "starting point" x and the "target point" y in an interval growing linearly with time t. In the original statements, the starting point satisfies $x = vt$ and the target point essentially always corresponds to the origin.

(B) Proposition 5.1(b) involves a perturbation by the end point (i.e., y changes to $y+h$), while the starting point is perturbed in the original statement.

In addition, [18, 20] always consider travelling waves going "from left to right", while for our purposes it is more suitable to work with waves going "from right to left". This difference is easily dealt with by mirroring the environment and we do not discuss it further.

Proving Proposition 5.1 thus requires checking that these two differences can be dealt with by the original arguments. In the following, we provide the arguments of the proof, and furthermore describe key locations where the arguments of [18, 20] have to be adapted. For auxiliary results from the original sources [18, 20] that can be employed without any further adaptation, we will just refer to the respective references.

We start by introducing certain logarithmic moment generating functions that are featured heavily in the proof. For $\eta < 0$ and $y > x$, let

$$\begin{aligned} \overline{L}^{\zeta}_{x,y}(\eta) &:= \frac{1}{y-x}\ln E_x\Big[\exp\Big\{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s\Big\}\Big] = \frac{1}{y-x}\ln\big(Z^{\zeta,\eta}_{x,y}\big), \\ L(\eta) &:= \mathbb{E}\big[\overline{L}^{\zeta}_{0,1}(\eta)\big]. \end{aligned} \tag{5.2}$$

(Note that the function $L(\eta)$ coincides with that in (4.7).) In order to control these functions, a simple but recurrently used generalisation of [20], Lemma A.1, is given by the following statement.

CLAIM 5.2.

(a) *For every $x < y$, the functions L and $\overline{L}^{\zeta}_{x,y}$ are infinitely differentiable on $(-\infty, 0)$ with*

$$\big(\overline{L}^{\zeta}_{x,y}\big)'(\eta) = \frac{1}{y-x} E^{\zeta,\eta}_x[H_y], \tag{5.3}$$

$$\big(\overline{L}^{\zeta}_{x,y}\big)''(\eta) = \frac{1}{y-x} \operatorname{Var}_{P^{\zeta,\eta}_x}(H_y) \tag{5.4}$$

(*where $\operatorname{Var}_{P^{\zeta,\eta}_x}$ denotes the variance with respect to $P^{\zeta,\eta}_x$ defined in* (4.6)).

(b) *There exists a constant* C_3 *such that*, $\mathbb{P}$-*a.s.*,

$$-C_3 \le \inf_{\eta\in\triangle,|x-y|\ge 1}\{\overline{L}^{\zeta}_{x,y}(\eta), L(\eta)\} \le \sup_{\eta\in\triangle,|x-y|\ge 1}\{\overline{L}^{\zeta}_{x,y}(\eta), L(\eta)\} \le -C_3^{-1}, \tag{5.5}$$

and, *for* $n \in \{1, 2\}$, *the nth derivatives satisfy*

$$\begin{aligned} C_3^{-1} &\le \inf_{\eta\in\triangle,|x-y|\ge 1}\{(\overline{L}^{\zeta}_{x,y})^{(n)}(\eta), L^{(n)}(\eta)\} \\ &\le \sup_{\eta\in\triangle,|x-y|\ge 1}\{(\overline{L}^{\zeta}_{x,y})^{(n)}(\eta), L^{(n)}(\eta)\} \le C_3. \end{aligned} \tag{5.6}$$

Since upon replacing the index x by indices x, y, the proof of this claim follows that of [20], Lemma A.1(b), verbatim, we omit it here. (In particular, it only uses the upper and lower bounds of the potential ζ, and otherwise is independent of it.)

To prove Proposition 5.1, we are now interested in the (random) tilting parameter $\eta^{\zeta}_{x,y}(v)$ for which the mean speed of a particle on its way from x to y under the tilted measure $P^{\zeta,\eta}_x$ is precisely v, that is,

$$E^{\zeta,\eta^{\zeta}_{x,y}(v)}_x[H_y] = \frac{y-x}{v}, \quad v > 0, x < y. \tag{5.7}$$

(If no such parameter exists, we set $\eta^{\zeta}_{x,y}(v) = 0$.) The next lemma shows that, for $y - x$ large, a unique $\eta^{\zeta}_{x,y}(v)$ satisfying (5.7) exists with high probability and that it is close to $\overline{\eta}(v)$ (recall (4.8)). It is an extension of Lemma 2.5 of [20] and it is the first step on the way to dealing with the difference (A) in the above list.

LEMMA 5.3.

(a) *For every* $A > 1$, *there exists a* $\mathbb{P}$-*a.s. finite random variable* $\mathcal{N} = \mathcal{N}(A)$ *such that for all* $v \in V$ *and* $x < y \in \mathbb{R}$ *such that* $y - x \ge \mathcal{N}$ *and* $|x|, |y| \le A(y - x)$, *the solution* $\eta^{\zeta}_{x,y}(v)$ *to* (5.7) *exists and satisfies* $\eta^{\zeta}_{x,y}(v) \in \triangle$.

(b) *For each* $q \in \mathbb{N}$, *there exists a constant* $C_4 = C_4(q) \in (0, \infty)$ *such that for all* $n \ge C_4$,

$$\mathbb{P}\Bigl(\sup_{v\in V}\sup_{x\in[-n,-n+1]}\sup_{y\in[0,1]}|\eta^{\zeta}_{x,y}(v) - \overline{\eta}(v)| \ge C_4\sqrt{\frac{\ln n}{n}}\Bigr) \le C_4 n^{-q}. \tag{5.8}$$

PROOF (OUTLINE). The proof follows the strategy of [20], Lemma 2.5: Item (a) follows directly from (b), using the Borel–Cantelli lemma and (5.1), with the help of an additional union bound to take care of the uniformity in y.

For the proof of (b), we remark that in [20], Lemma 2.5(b), it was shown that for all $n \ge C$,

$$\mathbb{P}\Bigl(\sup_{v\in V}\sup_{x\in[-n,-n+1]}|\eta^{\zeta}_{x,0}(v) - \overline{\eta}(v)| \ge C\sqrt{\frac{\ln n}{n}}\Bigr) \le C n^{-q}. \tag{5.9}$$

Hence, in order to derive (5.8), compared to (5.9), we now have to include the additional supremum over $y \in [0, 1]$. We first observe that replacing n by $n + 1$ in (5.9) and using the shift invariance of ζ implies that (5.9) also holds when $\eta^{\zeta}_{x,0}(v)$ is replaced by $\eta^{\zeta}_{x,1}(v)$. By (5.3) and (5.7), $\eta^{\zeta}_{x,y}(v)$ solves

$$\overline{L}'_{x,y}(\eta^{\zeta}_{x,y}(v)) = v^{-1}. \tag{5.10}$$

From (5.3), it follows that $y \mapsto (y - x)(\overline{L}^{\zeta}_{x,y})'(\eta)$ is increasing. Hence, for $y \in [0, 1]$,

$$(0 - x)\overline{L}'_{x,0}(\eta^{\zeta}_{x,y}(v)) \le (y - x)\overline{L}'_{x,y}(\eta^{\zeta}_{x,y}(v)) \le (1 - x)\overline{L}'_{x,1}(\eta^{\zeta}_{x,y}(v)). \tag{5.11}$$

The first inequality and (5.10) then imply

$$\overline{L}'_{x,0}(\eta^{\zeta}_{x,y}(v)) \leq \frac{y-x}{v(0-x)} \leq \frac{1}{v}\left(1+\frac{1}{|x|}\right).$$

By (5.6), $\overline{L}'_{x,0}$ is increasing and, on $\triangle$, its derivative is bounded from below by C_3^{-1}. Hence, on the event that $\eta^{\zeta}_{x,0}(v)$ is close to $\overline{\eta}(v)$ (which is typical for n large, by (5.9)), this implies

$$\eta^{\zeta}_{x,y}(v) \leq \eta^{\zeta}_{x,0}(v) + \frac{C_3}{v|x|}.$$

Similarly, the second inequality in (5.11) implies

$$\eta^{\zeta}_{x,y}(v) \geq \eta^{\zeta}_{x,1}(v) - \frac{C_3}{v|x|}.$$

Combining these two inequalities with (5.9) (and its version for $\eta^{\zeta}_{x,1}(v)$, as observed above) then yields (5.8) with $C_4 = 2C$ for n large, and by adjusting C_4 for all $n \in \mathbb{N}$. □

We next adapt Lemma 2.7 of [20], which provides a spatial perturbation result for the tilting parameters $\eta^{\zeta}_{x,y}(v)$ introduced in (5.7).

LEMMA 5.4. *For $A > 1$, let $\mathcal{N} = \mathcal{N}(A)$ be as in Lemma 5.3. There exists a constant C_5 such that for all $x, y \in \mathbb{R}$ with $y - x \geq \mathcal{N}$ and $|x|, |y| \leq A(y-x)$, $v \in V$ and $h \in [1, y-x]$, we have*

$$\left|\eta^{\zeta}_{x,y}(v) - \eta^{\zeta}_{x,y+h}(v)\right| \leq \frac{C_5 h}{y-x}. \tag{5.12}$$

PROOF (OUTLINE). The strategy for proving Lemma 5.4 is as follows: By Lemma 5.3, $\eta^{\zeta}_{x,y}(v), \eta^{\zeta}_{x,y+h}(v) \in \triangle$ for all x, y, v, h as in the assumptions. In particular, this means that $\eta^{\zeta}_{x,y}(v)$ and $\eta^{\zeta}_{x,y+h}(v)$ are implicitly defined via (5.7) or, equivalently, (5.10). Therefore, the proof then proceeds by showing that there exists C_6 such that

$$\sup_{\eta\in\triangle}\left|(\overline{L}^{\zeta}_{x,y+h})'(\eta) - (\overline{L}^{\zeta}_{x,y})'(\eta)\right| \leq C_6\frac{h}{y-x}, \tag{5.13}$$

which will complete the proof, since again one can use the regularity of the functions $\overline{L}^{\zeta}_{x,y}$ and their derivatives to deduce the closeness of their arguments from their function values being close.

It remains to establish (5.13). For this purpose, note that using (5.3) and the strong Markov property at time H_y, one rewrites

$$(\overline{L}^{\zeta}_{x,y+h})'(\eta) - (\overline{L}^{\zeta}_{x,y})'(\eta) = -\frac{h}{y-x+h}(\overline{L}^{\zeta}_{x,y})'(\eta) + \frac{h}{y-x+h}(\overline{L}^{\zeta}_{y,y+h})'(\eta).$$

Claim (5.13) then readily follows using the bounds for $(\overline{L}^{\zeta}_{x,y})'$ from (5.6). □

Next, we introduce two auxiliary processes, which we later relate to the expressions appearing in Proposition 5.1. We consider, for $x < y \in \mathbb{R}$ and $v > 0$, the quantities

$$\begin{aligned} Y_v^{\approx}(x,y) &:= E_x\left[e^{\int_0^{H_y}\zeta(X_s)\,\mathrm{d}s}; H_y \in \left[\frac{y-x}{v} - K, \frac{y-x}{v}\right]\right], \\ Y_v^{>}(x,y) &:= E_x\left[e^{\int_0^{H_y}\zeta(X_s)\,\mathrm{d}s}; H_y < \frac{y-x}{v} - K\right], \end{aligned} \tag{5.14}$$

where $K > 0$ is a large constant that will be fixed later on in (5.22) below. For this suitable choice of K, the quantities $Y_v^{\approx}(x, y)$ and $Y_v^{<}(x, y)$ are comparable uniformly in the admissible choices of x and y. This result is a partial extension of [20], Proposition 3.5.

LEMMA 5.5. *For $A > 1$, let $\mathcal{N} = \mathcal{N}(A)$ be as in Lemma 5.3. Then there exists a constant C_7 such that for all $v \in V$ and all $x < y \in \mathbb{R}$ such that $y - x \geq \mathcal{N}$ as well as $|x|, |y| \leq A(y - x)$, we have*

$$\frac{Y_v^{\approx}(x, y)}{Y_v^{<}(x, y)} \in [C_7^{-1}, C_7]. \tag{5.15}$$

PROOF. The proof of this lemma contains a computation that is essential for a step in the proof of Lemma 6.1, and is also featured in Section 6 below. We assume that x, y satisfy the assumptions of the lemma, and in order to keep the notation simple, we in addition assume that $x, y \in \mathbb{Z}$ (we refer to [20], Section 1.9, for the technical issue and notation of how to deal with the case of noninteger x and y). We write $\eta^* := \eta_{x,y}^{\zeta}(v)$ for conciseness of notation in the following and define

$$\sigma^* = \sigma_{x,y}^{\zeta}(v) := |\eta^*| \sqrt{\mathrm{Var}_{P_x^{\zeta,\eta^*}}(H_y)}. \tag{5.16}$$

We observe that $\mathbb{P}$-a.s., uniformly in $v \in V$,

$$C_8 \sqrt{y - x} \leq \sigma_{x,y}^{\zeta}(v) \leq C_8 \sqrt{y - x}. \tag{5.17}$$

Indeed, this follows directly from (5.4) and (5.6), with C_8 depending only on C_3 and $\triangle$.

Let further $\tau_z = H_z - H_{z-1}$, $z \in [x + 1, y] \cap \mathbb{Z}$ and let $\widehat{\tau}_z := \tau_z - E_x^{\zeta,\eta^*}[\tau_z]$. Then, by the definition of η^*, for x, y satisfying the assumptions of Lemma 5.3, we have $E_x^{\zeta,\eta^*}[H_y] = \frac{y-x}{v}$. With this notation, a straightforward computation yields

$$\begin{aligned} & Y_v^{\approx}(x, y) \\ & = E_x\Bigl[e^{\int_0^{H_y}(\zeta(X_s)+\eta^*)\,\mathrm{d}s} e^{-\eta^* \sum_{z=x+1}^{y} \widehat{\tau}_z}; \sum_{i=x+1}^{y} \widehat{\tau}_z \in [-K, 0]\Bigr] e^{-(y-x)\eta^*/v} \\ & = E_x^{\zeta,\eta^*}\Bigl[e^{-\sigma^* \frac{\eta^*}{\sigma^*} \sum_{z=x+1}^{y} \widehat{\tau}_z}; \frac{\eta^*}{\sigma^*} \sum_{z=x+1}^{y} \widehat{\tau}_z \in \Bigl[0, -\frac{K\eta^*}{\sigma^*}\Bigr]\Bigr] e^{-(y-x)(\frac{\eta^*}{v} - \overline{L}_{x,y}^{\zeta}(\eta^*))}. \end{aligned} \tag{5.18}$$

Defining $\mu_{x,y}^{\zeta,\eta^*}$ to be the distribution of $\frac{\eta^*}{\sigma^*} \sum_{z=x+1}^{y} \widehat{\tau}_z$ under P_x^{ζ,η^*}, this implies

$$Y_v^{\approx}(x, y) = e^{-(y-x)(\frac{\eta^*}{v} - \overline{L}_{x,y}^{\zeta}(\eta^*))} \int_0^{\frac{-K\eta^*}{\sigma^*}} e^{-\sigma^* u} \mu_{x,y}^{\zeta,\eta^*}(\mathrm{d}u). \tag{5.19}$$

A completely analogous computation then shows that

$$Y_v^{<}(x, y) = e^{-(y-x)(\frac{\eta^*}{v} - \overline{L}_{x,y}^{\zeta}(\eta^*))} \int_{\frac{-K\eta^*}{\sigma^*}}^{\infty} e^{-\sigma^* u} \mu_{x,y}^{\zeta,\eta^*}(\mathrm{d}u). \tag{5.20}$$

The upshot of these computations is that under P_x^{ζ,η^*}, the random variables $\widehat{\tau}_z$, $z = x + 1, \dots, y$, are centred, independent, have uniform exponential moments and $\mu_{x,y}^{\zeta,\eta^*}$ has unit variance. As a consequence, we can uniformly approximate $\mu_{x,y}^{\zeta,\eta^*}$ by the standard Gaussian measure Φ, and the integrals appearing on the right-hand side of (5.19) and (5.20) are $\mathbb{P}$-a.s. both of order $(y - x)^{-1/2}$, uniformly in the ζ and $v \in V$ under consideration, and for all x, y satisfying the assumptions of Lemma 5.3. More precisely, the conditions of the error estimate ([8], Theorem 13.3) for normal approximations are satisfied and we can hence apply [8], (13.43), to deduce that

$$\sup_{\mathcal{C}} \left|\mu_{x,y}^{\zeta,\eta^*}(\mathcal{C}) - \Phi(\mathcal{C})\right| \le C_9 (y-x)^{-1/2}, \tag{5.21}$$

where the supremum is over all intervals in $\mathbb{R}$ and the constant $C_9 > 0$ only depends on the uniform bound of the exponential moments of the $\widehat{\tau}_z$'s. Now, due to (5.17), we can choose $K > 0$ large enough so that for some constants $C_9 < C_{10} = C_{10}(K) < C_{11} = C_{11}(K)$, for all x, y with $y - x \ge \mathcal{N}$ and $v \in V$ we have

$$C_{10}(y-x)^{-1/2} \le \Phi\big([0, -K\eta^*/\sigma]\big) \le C_{11}(y-x)^{-1/2}, \tag{5.22}$$

and thus infer

$$(C_{10} - C_9)(y-x)^{-1/2} \le \mu_{x,y}^{\zeta,\eta^*}\big([0, -K\eta^*/\sigma]\big) \le (C_{11} + C_9)(y-x)^{-1/2}.$$

Plugging these estimates into (5.19) and (5.20), we can then show that the integrals in these displays are both of the same order $(y-x)^{-1/2}$, following exactly the same steps as in [20], Lemma 3.6. $\square$

Lemma 5.5 has an important corollary allowing to approximate the Feynman–Kac formula for the PAM (cf. (3.6)) by expressions involving $Y_v^{\approx}(x, y)$. This approximation will also be used in Section 6 below. Its proof is inspired by that of Lemma 3.7 in [20].

LEMMA 5.6. *For each $A > 1$, with $\mathcal{N} = \mathcal{N}(A)$ as in Lemma 5.3, there exists a constant $C_{12} = C_{12}(K) \in (1, \infty)$ such that for all $t \in (0, \infty)$ and all $x < y \in \mathbb{R}$ such that $y - x \ge \mathcal{N}$, $|x|, |y| \le A(y-x)$ and $\frac{y-x}{t} \in V$,*

$$C_{12}^{-1} Y_v^{\approx}(x, y) \le E_x\big[e^{\int_0^t \zeta(X_s)\,\mathrm{d}s}; X_t \ge y\big] \le C_{12} Y_v^{\approx}(x, y). \tag{5.23}$$

PROOF. Since for X_t starting in x, we have $\{X_t \ge y\} \subset \{H_y \le t\}$ and $\zeta \le 0$, and we get $E_x[e^{\int_0^t \zeta(X_s)\,\mathrm{d}s}; X_t \le y] \le (1 + C_7) Y_v^{\approx}(x, y)$ by Lemma 5.5, and thus the last inequality in (5.23) is obtained.

To show the first inequality, we define for $\delta > 0$ and $y \in \mathbb{R}$ the random functions

$$p_y(s) := E_y\big[e^{\int_0^s \zeta(X_r)\,\mathrm{d}r}; X_s \in [y, y+\delta]\big],$$

which, almost surely, for all $s \in [0, K]$, are bounded from below by a deterministic constant $c_1 = c_1(K, \delta) \in (0, 1)$. Using the strong Markov property at H_y, we finally get

$$\begin{aligned} Y_v^{\approx}(x, y) &= E_x\big[e^{\int_0^{H_y} \zeta(X_r)\,\mathrm{d}r}; H_y \in [t-K, t]\big] \\ &\le c_1^{-1} E_x\big[e^{\int_0^{H_y} \zeta(X_r)\,\mathrm{d}r} p_y(t - H_y); H_y \in [t-K, t]\big] \\ &\le c_1^{-1} E_x\big[e^{\int_0^{H_y} \zeta(X_r)\,\mathrm{d}r} p_y(t - H_y)\big] \\ &= c_1^{-1} E_x\big[e^{\int_0^t \zeta(X_r)\,\mathrm{d}r}; X_t \in [y, y+\delta]\big] \end{aligned}$$

and the claim follows by choosing $C_{12} := c_1^{-1} \vee (1 + C_7)$. $\square$

With this, we have made all the necessary extensions to the results in [18, 20], which are needed in order to accommodate for the differences outlined in (A) and (B) and are equipped to show Proposition 5.1.

PROOF OF PROPOSITION 5.1. We denote $v := (y-x)/t$, $v' := (y-x)/(t+h)$ and observe that by Lemma 5.6, for x, y, t and h as in the statement, and by choosing $\mathcal{T}_1$ sufficiently large so that $y - x \geq \mathcal{N}$,

$$\frac{E_x[e^{\int_0^{t+h}\xi(X_s)\,\mathrm{d}s}; X_{t+h} \geq y]}{E_x[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y]} \leq C_{12}^2 \frac{Y_{v'}^{\approx}(x,y)}{Y_v^{\approx}(x,y)}. \tag{5.24}$$

The fraction on the right-hand side can be rewritten with the help of (5.19). Using also the fact that the integral appearing in (5.19) is of order $(y-x)^{-1/2}$ uniformly in $v \in V$ and $y - x \geq \mathcal{N}$, as explained at the end of the proof of Lemma 5.5, we obtain

$$\frac{Y_{v'}^{\approx}(x,y)}{Y_v^{\approx}(x,y)} \leq C_{13} \frac{\exp\{-(y-x)(\frac{\eta_{x,y}^{\zeta}(v')}{v'} - \overline{L}_{x,y}^{\zeta}(\eta_{x,y}^{\zeta}(v')))\}}{\exp\{-(y-x)(\frac{\eta_{x,y}^{\zeta}(v)}{v} - \overline{L}_{x,y}^{\zeta}(\eta_{x,y}^{\zeta}(v)))\}}, \tag{5.25}$$

for some finite constant $C_{13} = C_{13}(K)$. Denoting for any $\eta < 0$,

$$S_{x,y}^{\zeta,v}(\eta) := (y-x)\left(\frac{\eta}{v} - \overline{L}_{x,y}^{\zeta}(\eta)\right), \tag{5.26}$$

the logarithm of the fraction on the right-hand side of (5.25) can be written as

$$\left(S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v)) - S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v'))\right) + \left(S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v')) - S_{x,y}^{\zeta,v'}(\eta_{x,y}^{\zeta}(v'))\right) \tag{5.27}$$

Recalling the definitions of v and v', the second summand in (5.27) satisfies

$$\left(S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v')) - S_{x,y}^{\zeta,v'}(\eta_{x,y}^{\zeta}(v'))\right) = -h\eta_{x,y}^{\zeta}(v') \leq C_{14}h, \tag{5.28}$$

for some finite constant C_{14}, since $\eta_{x,y}^{\zeta}(v') \in \triangle$ for the x, y, v' under consideration due to Lemma 5.3(a). Moreover, the absolute value of the first summand in (5.27) can be upper bounded by $ch^2/t \ll h$ uniformly for x, y, t and h under consideration, exactly as in the paragraph containing [20], (3.39), (this proof uses again only estimates that are uniform in ζ). This completes the proof of part (a).

The proof of part (b) follows the lines of the proof of the spatial perturbation result, Lemma 4.1 in [18]. Indeed, using the same reasoning as in (5.24)–(5.27), now choosing $v := (y-x)/t$ and $v' := (y+h-x)/t$, where x, y, t and h are as in the statement and $\mathcal{T}_2$ is assumed to be sufficiently large so that $y - x \geq \mathcal{N}$, we infer that

$$\frac{E_x[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y+h]}{E_x[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y]} \leq C_{15} \frac{Y_{v'}^{\approx}(x,y+h)}{Y_v^{\approx}(x,y)}, \tag{5.29}$$

as well as

$$\begin{aligned} \ln \frac{Y_{v'}^{\approx}(x,y+h)}{Y_v^{\approx}(x,y)} &\leq \left(S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v)) - S_{x,y+h}^{\zeta,v'}(\eta_{x,y}^{\zeta}(v))\right) \\ &\quad + \left(S_{x,y+h}^{\zeta,v'}(\eta_{x,y}^{\zeta}(v)) - S_{x,y+h}^{\zeta,v'}(\eta_{x,y+h}^{\zeta}(v'))\right). \end{aligned} \tag{5.30}$$

By (5.2) and (5.26), the first summand on the right-hand side of (5.30) (which differs slightly from the corresponding one in [18], due to item (B) above) satisfies

$$\begin{aligned} &\left|S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v)) - S_{x,y+h}^{\zeta,v'}(\eta_{x,y}^{\zeta}(v))\right| \\ &\quad = \left|\ln E_x\left[e^{\int_0^{H_{y+h}}(\zeta(X_s)+\eta_{x,y}^{\zeta}(v))\,\mathrm{d}s}\right] - \ln E_x\left[e^{\int_0^{H_y}(\zeta(X_s)+\eta_{x,y}^{\zeta}(v))\,\mathrm{d}s}\right]\right| \\ &\quad = \left|\ln E_y\left[e^{\int_0^{H_{y+h}}(\zeta(X_s)+\eta_{x,y}^{\zeta}(v))\,\mathrm{d}s}\right]\right| \\ &\quad \leq h\sqrt{2(\mathtt{es} - \mathtt{ei} + |\eta_{x,y}^{\zeta}(v)|)} \leq C_{16}h, \end{aligned} \tag{5.31}$$

where, in the second equality, we applied the strong Markov property at time H_y, and used (4.18) for the final inequality and $C_{16} = C_{16}(V, \triangle, \mathtt{ei}, \mathtt{es})$ some finite constant.

The second summand on the right-hand side of (5.30) is upper bounded by $Ch^2/t \ll h$ and is thus negligible. This can be proved exactly as in [18], (4.13)–(4.16). Besides [20], Lemma 2.7, which we already extended in Lemma 5.4, this proof again only uses uniform estimates, and thus does not require any modification. This completes the proof of the proposition. $\square$

6. Dependence of solutions to the F-KPP equation on the initial condition. In this section, we prove the key technical lemma, Lemma 6.1 below, which formalises inequalities (2.15) and (2.16), and which provides the right ordering of two solutions to (F-KPP) with different initial conditions. Its proof is based on a careful examination of the Feynman–Kac representations of these solutions, using all the tools that were introduced in previous sections.

To state the lemma, we introduce two auxiliary speeds,

$$v_1 := \sqrt{2(\mathtt{es}+1)} \quad \text{and} \tag{6.1}$$

$$v_2 := \inf\{v > v_1 + 1 : |\overline{\eta}(v)| \geq 2v_1^2 + 2\}, \tag{6.2}$$

where $\overline{\eta}(v)$ was defined above (4.8); note that display (4.8) also ensures that v_2 is finite. By comparing the BBMRE with the BBM with constant branching rate $\mathtt{es}$, for which the speed of the maximum is $\sqrt{2\mathtt{es}}$, we obtain

$$v_0 < v_1 < v_2.$$

Recall the notation w^y from below (2.11).

LEMMA 6.1. *For each $u > 0$ and each $v > v_2$, there exist $\Delta_0 = \Delta_0(u, v) > 0$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T} = \mathcal{T}(u, v)$, such that $\mathbb{P}$-a.s., for all $\Delta > \Delta_0$, $y \in [0, vt]$ and $t \geq \mathcal{T}$,*

$$w^y(t, y - vt) \geq w^{y+\Delta}(t + u, y - vt). \tag{6.3}$$

PROOF. We start by upper bounding the right-hand side of (6.3). By the Feynman–Kac representation (3.7) and the fact that $\sup_{w\in[0,1]} \widetilde{F}(w) = 1$ (cf. Proposition 3.3), it follows for any $\Delta > 0$ that

$$\begin{aligned} w^{y+\Delta}(t+u, y-vt) &= E_{y-vt}\big[e^{\int_0^{t+u}\xi(X_s)\widetilde{F}(w^{y+\Delta}(t+u-s,X_s))\,\mathrm{d}s}; X_{t+u} \geq y+\Delta\big] \\ &\leq E_{y-vt}\big[e^{\int_0^{t+u}\xi(X_s)\,\mathrm{d}s}; X_{t+u} \geq y+\Delta\big]. \end{aligned} \tag{6.4}$$

To the right-hand side of (6.4), we now successively apply both parts of the perturbation result of Proposition 5.1 (with V sufficiently large, as explained before Proposition 5.1 and $A = 2v$). In order to apply them, we let $t \geq u \vee \mathcal{T}_1 \vee \mathcal{T}_2 =: \mathcal{T}$, where $\mathcal{T}_1, \mathcal{T}_2$ are the $\mathbb{P}$-a.s. finite random variables appearing in the statement of the perturbation lemma. For such t, we then obtain

$$\begin{aligned} w^{y+\Delta}(t+u, y-vt) &\leq C_1 e^{C_1 u} E_{y-vt}\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y+\Delta\big] \\ &\leq C_1 C_2 e^{C_1 u - \Delta/C_2} E_{y-vt}\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y\big], \end{aligned} \tag{6.5}$$

which is our first intermediate inequality.

Let us now turn our focus to bounding the left-hand side of (6.3) from below. By the Feynman–Kac representation (3.7),

$$w^y(t, y - vt) = E_{y-vt}\Big[\exp\Big\{\int_0^t \xi(X_r)\widetilde{F}\big(w^y(t-r, X_r)\big)\,\mathrm{d}r\Big\}; X_t \ge y\Big]. \tag{6.6}$$

We now claim that $\widetilde{F}$ satisfies,

$$\widetilde{F}(w) = F(w)/w \ge 1 - \frac{1}{2}(\mu_2 - 2)w, \quad w \in [0, 1]. \tag{6.7}$$

Indeed, by (3.1) and the normalisation (3.4) of Remark 3.2, the nonlinearity F of (F-KPP) satisfies $F(0) = 0$ and $F'(0) = 1$. In addition, by (3.5), $F'' \ge -\mu_2 + 2$ on $[0, 1]$. Therefore, by a first-order Taylor approximation with Lagrange remainder,

$$F(w) \ge w + \frac{1}{2}\inf_{w^* \in [0,1]} F''(w^*)w^2 = w - \frac{1}{2}(\mu_2 - 2)w^2,$$

from which (6.7) directly follows.

Plugging (6.7) into (6.6) and using the uniform boundedness (2.4) from Assumption 1, we arrive at

$$w^y(t, y - vt) \ge E_{y-vt}\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}\, e^{-\frac{\mathrm{es}}{2}(\mu_2-2)\int_0^t w^y(t-s,X_s)\,\mathrm{d}s}; X_t \ge y\big]. \tag{6.8}$$

In order to obtain a suitable control of the second exponential factor in (6.8), we construct an event restricted to which the second exponential is bounded from below in a suitable way. For this purpose, recall the definition of v_1 from (6.1), and introduce for given t, y the *moving boundary*

$$\beta_{y,t}(s) := y - v_1(t - s), \quad s \in [0, t]. \tag{6.9}$$

By $\mathcal{T}_{y,t} := \inf\{s \ge 0 : X_s = \beta_{y,t}(s)\}$, we denote the first hitting time of $\beta_{y,t}$ by a Brownian motion started at $y - vt$.

We claim that for $K > 1 \vee v_1^{-2}$ to be fixed later, on the good event

$$\mathcal{G} := \{\mathcal{T}_{y,t} \in [t - K, t]\} \tag{6.10}$$

it holds that

$$\int_0^{t-K} w^y(t - s, X_s)\,\mathrm{d}s \le 1; \tag{6.11}$$

see Figure 3 for an illustration. Indeed, note that using again the Feynman–Kac representation (3.7) as well as the uniform boundedness (2.4) of Assumption 1, in combination with the fact that $\sup_{w\in[0,1]} \widetilde{F}(w) \le 1$ once more, it holds that

$$\begin{aligned} w^y(t - s, X_s) &\le E_{X_s}\big[e^{\int_0^{t-s} \xi(\widetilde{X}_r)\,\mathrm{d}r}; \widetilde{X}_{t-s} \ge y\big] \\ &\le e^{\mathrm{es}(t-s)} P_{X_s}(\widetilde{X}_{t-s} \ge y), \end{aligned}$$

where we write $\widetilde{X}$ for an independent Brownian motion started at X_s in order to avoid confusion between the two processes. On $\mathcal{G}$, one has that $X_s \le y - v_1(t - s)$ for $s \in [0, t - K]$. Hence, by a straightforward coupling argument, on $\mathcal{G}$ we have

$$P_{X_s}(\widetilde{X}_{t-s} \ge y) \le P_0\big(\widetilde{X}_{t-s} \ge v_1(t - s)\big) = P(Z \ge v_1\sqrt{t - s}),$$

where Z is a standard Gaussian random variable. Using this in combination with a standard Gaussian bound (see, e.g., [2], (1.2.2)) and taking advantage of the fact that $v_1\sqrt{(t - s)} \ge$

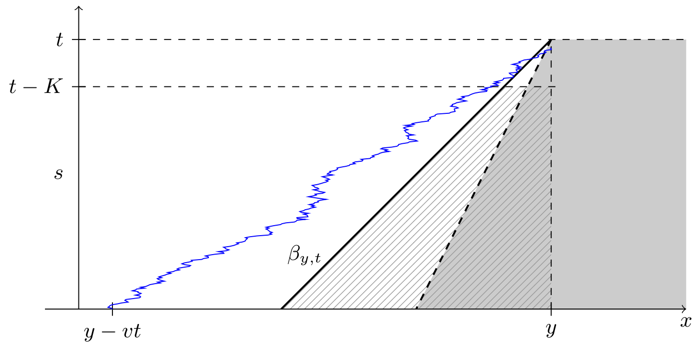

FIG. 3. *Sketch of a trajectory of the Brownian motion $(X_s)_{s\geq 0}$, started at $y-vt$, up until the hitting time H_y of y, which realises the good event $\mathcal{G}$. This trajectory does not hit the moving barrier $\beta_{y,t}(s)$ (thick solid line) in the time interval $[0,t-K]$, and thus avoids the dashed region. The function $w^y(t-s,\cdot)$ is close to 1 in the grey region, close to 0 in its complement and changes its value from 0 to 1 in the vicinity of the thick dashed line whose slope is v_0.*

$v_1\sqrt{K}\geq 1$, it follows that on $\mathcal{G}$ we can upper bound

$$\begin{aligned}\int_0^{t-K} w^y(t-s,X_s)\,\mathrm{d}s &\leq \int_0^{t-K} e^{\mathtt{es}(t-s)}P(Z\geq v_1\sqrt{t-s})\,\mathrm{d}s\\ &\leq \frac{1}{\sqrt{2\pi}}\int_0^{t-K} e^{-(v_1^2/2-\mathtt{es})(t-s)}\,\mathrm{d}s\\ &= \frac{1}{\sqrt{2\pi}}\int_K^t e^{-(v_1^2/2-\mathtt{es})z}\,\mathrm{d}z\\ &\leq \frac{1}{\sqrt{2\pi}(v_1^2/2-\mathtt{es})}e^{-K(v_1^2/2-\mathtt{es})}\leq 1,\end{aligned}\tag{6.12}$$

where in the last inequality we used $v_1^2/2-\mathtt{es}=1$, which holds by (6.1). This proves (6.11).

Coming back to the task of finding a lower bound for the right-hand side of (6.8), we infer from the above discussion that on $\mathcal{G}$ we can use (6.11) to bound the second exponential factor on the right-hand side of (6.8) by

$$\begin{aligned}e^{-\frac{\mathtt{es}}{2}(\mu_2-2)\int_0^t w^y(t-s,X_s)\,\mathrm{d}s} &\geq e^{-\frac{\mathtt{es}}{2}(\mu_2-2)(1+\int_{t-K}^t w^y(t-s,X_s)\,\mathrm{d}s)}\\ &\geq e^{-\frac{\mathtt{es}}{2}(\mu_2-2)(1+K)},\end{aligned}\tag{6.13}$$

where in the last inequality we used that $0\leq w^y(s,y)\leq 1$ uniformly for all $(s,y)\in[0,\infty)\times\mathbb{R}$. Consequently, by restricting the expectation on the right-hand side of (6.8) to $\mathcal{G}$, it follows by (6.13) that whenever $v>v_1$, then

$$w^y(t,y-vt)\geq e^{-\frac{\mathtt{es}}{2}(\mu_2-2)(1+K)}E_{y-vt}\big[e^{\int_0^t\xi(X_s)\,\mathrm{d}s};X_t\geq y,\ \mathcal{G}\big].\tag{6.14}$$

This is our second intermediate inequality.

In order to complete the proof of (6.3), we need to compare the expectations on the right-hand side of (6.5) and on the right-hand side of (6.14). This is the purpose of the following lemma.

LEMMA 6.2. *Let v_2 be as in (6.2) and $\mathcal{G}$ as in (6.10). Then, for every $v>v_2$ there exist constants $K=K(v)$ and $\widetilde{C}=\widetilde{C}(v)\in(0,\infty)$ such that $\mathbb{P}$-a.s., for all t large enough and all $y\in[0,vt]$, one has*

$$E_{y-vt}\big[e^{\int_0^t\xi(X_s)\,\mathrm{d}s};X_t\geq y\big]\leq\widetilde{C}E_{y-vt}\big[e^{\int_0^t\xi(X_s)\,\mathrm{d}s};X_t\geq y,\mathcal{G}\big].\tag{6.15}$$

We postpone the proof of Lemma 6.2 and complete the proof of Lemma 6.1 first. By combining the lower bound (6.14), the upper bound (6.5) and Lemma 6.2, we obtain

$$\begin{aligned} &w^y(t, y - vt) - w^{y+\Delta}(t + u, y - vt) \\ &\quad \geq \big(e^{-\frac{\mathtt{es}}{2}(\mu_2-2)(K+1)} - \widetilde{C} C_1 C_2 e^{C_1 u - \Delta/C_1}\big) E_{y-vt}\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y, \mathcal{G}\big]. \end{aligned}$$

For every Δ satisfying

$$\Delta \geq \Delta_0 := C_2\Big(C_1 u + \frac{\mathtt{es}}{2}(\mu_2 - 2)(K + 1) + \ln(\widetilde{C} C_1 C_2)\Big),$$

the right-hand side is positive, which proves (6.3), and thus the lemma. □

PROOF OF LEMMA 6.2. To prove the lemma, we use the machinery of tilted measures as introduced in Section 4. We recall the notation $\zeta = \xi - \mathtt{es}$ from (4.1) and observe that, by multiplying both sides of (6.15) by $e^{-\mathtt{es}t}$, it is sufficient to show (6.15) with ζ in place of ξ.

We start by proving an upper bound for the left-hand side of (6.15) in terms of tilted measures. By Lemma 5.6, there exist constants $C, L < \infty$ such that for any $\eta < 0$, for t large enough uniformly in $y \in [0, vt]$ it holds that

$$\begin{aligned} E_{y-vt}\big[e^{\int_0^t \zeta(X_s)\,\mathrm{d}s}; X_t \geq y\big] &\leq C E_{y-vt}\big[e^{\int_0^{H_y} \zeta(X_s)\,\mathrm{d}s}\ H_y \in [t - L, t]\big] \\ &\leq C e^{-\eta t} Z_{y-vt,y}^{\zeta,\eta} P_{y-vt}^{\zeta,\eta}\big(H_y \in [t - L, t]\big). \end{aligned} \tag{6.16}$$

In the next step, we bound the expression appearing on the right-hand side of (6.15) from below. To this end, let $p_y^{\zeta,\eta}(t) := P_y^{\zeta,\eta}(X_t \geq y)$. Using the strong Markov property, we obtain

$$\begin{aligned} &E_{y-vt}\big[e^{\int_0^t \zeta(X_s)\,\mathrm{d}s}; X_t \geq y, \mathcal{T}_{y,t} \geq t - K\big] \\ &\quad\geq e^{-(\mathtt{es}-\mathtt{ei})K} E_{y-vt}\big[e^{\int_0^{H_y} \zeta(X_s)\,\mathrm{d}s}; H_y \in [t - K, t], X_t \geq y, \mathcal{T}_{y,t} \geq t - K\big] \\ &\quad\geq e^{-(\mathtt{es}-\mathtt{ei}-\eta)K} e^{-\eta t} \\ &\qquad\times E_{y-vt}\big[e^{\int_0^{H_y} (\zeta(X_s)+\eta)\,\mathrm{d}s}; H_y \in [t - K, t], X_t \geq y, \mathcal{T}_{y,t} \geq t - K\big] \\ &\quad= e^{-(\mathtt{es}-\mathtt{ei}-\eta)K} e^{-\eta t} Z_{y-vt,y}^{\zeta,\eta} \\ &\qquad\times E_{y-vt}^{\zeta,\eta}\big[p_y^{\zeta,\eta}(t - H_y), H_y \in [t - K, t], \mathcal{T}_{y,t} \geq t - K\big] \\ &\quad\geq \frac{1}{2} e^{-(\mathtt{es}-\mathtt{ei}-\eta)K} e^{-\eta t} Z_{y-vt,y}^{\zeta,\eta} P_{y-vt}^{\zeta,\eta}\big(H_y \in [t - K, t], \mathcal{T}_{y,t} \geq t - K\big), \end{aligned} \tag{6.17}$$

where in the last inequality we used Lemma 4.3 to infer that for any $\eta < 0$ and $s \geq 0$ one has $p_y^{\zeta,\eta}(s) \geq P_0^{\sqrt{2|\eta|}}(X_s \geq 0) \geq 1/2$.

In view of (6.16) and (6.17), in order to complete the proof of Lemma 6.2, it is sufficient to show that

$$P_{y-vt}^{\zeta,\eta}\big(H_y \in [t - L, t]\big) \leq C P_{y-vt}^{\zeta,\eta}\big(H_y \in [t - K, t], \mathcal{T}_{y,t} \geq t - K\big), \tag{6.18}$$

for some suitably chosen parameter η and constants C, K, L, $\mathbb{P}$-a.s. for all t large, uniformly in $y \in [0, vt]$.

To this end, we will need two further auxiliary lemmas. The first one will be used to upper bound the probability appearing on the right-hand side of (6.18), and also specifies the range of suitable η's.

LEMMA 6.3. *Let $\eta < 0$ be such that $\sqrt{2|\eta|} > v_1(1 + \frac{2L}{K})$, and let $0 < L < K$ be such that $L/K \leq 1/3$. Then $\mathbb{P}$-a.s. for every $y \in \mathbb{R}$ and $v > v_1$,*

$$P^{\zeta,\eta}_{y-vt}(H_y \leq t, \mathcal{T}_{y,t} \leq t - K) \leq 2P^{\zeta,\eta}_{y-vt}(H_y < t - L). \tag{6.19}$$

The second auxiliary lemma is a quantitative extension of a part of Proposition 3.5 of [20]. It states that under the tilted measure, if the tilting is not too strong, the probability of crossing a large interval in time t is comparable to the probability of crossing the same interval in time $t - L$.

LEMMA 6.4. *For every $v > v_c$, there is $c = c(v) < \infty$ such that for all L large enough and $\eta \in (\overline{\eta}(v) + \frac{c}{L}, 0)$, $\mathbb{P}$-a.s. for all t large enough and $|y| \leq 2vt$,*

$$P^{\zeta,\eta}_{y-vt}(H_y \leq t - L) \leq \frac{1}{4} P^{\zeta,\eta}_{y-vt}(H_y \leq t),$$

and as a consequence,

$$P^{\zeta,\eta}_{y-vt}(H_y \leq t - L) \leq \frac{1}{3} P^{\zeta,\eta}_{y-vt}(H_y \in (t - L, t]).$$

We postpone the proofs of these two lemmas to the end of the current section. We now come back to the proof of Lemma 6.2 and complete it by showing (6.18). To this end, we choose the parameters η, K and L in such a way that the previous two lemmas can be used simultaneously. More precisely, for a given $v \geq v_2$ we fix arbitrary η so that

$$\left|\overline{\eta}(v)\right| - 1 > |\eta| > 2v_1^2, \tag{6.20}$$

which is possible by the definition of v_2 in (6.2). Then we fix L as large as required in Lemma 6.4. Consequently, due to (6.20), the required assumptions on η are satisfied in our setting. Finally, we fix $K \geq 3L$ and observe that, in combination with (6.20), $\sqrt{2|\eta|} > 2v_1 \geq v_1(1 + \frac{2L}{K})$, so that the assumptions of Lemma 6.3 are satisfied also.

With this choice of constants, noting that $\{H_y \in [t - K, t], \mathcal{T}_{y,t} \geq t - K\} = \{H_y \leq t, \mathcal{T}_{y,t} \geq t - K\}$ (cf. Figure 3 also), the right-hand side of (6.18) satisfies

$$\begin{aligned} &P^{\zeta,\eta}_{y-vt}\big(H_y \in [t - K, t], \mathcal{T}_{y,t} \geq t - K\big) \\ &\quad = P^{\zeta,\eta}_{y-vt}(H_y \leq t) - P^{\zeta,\eta}_{y-vt}(H_y \leq t, \mathcal{T}_{y,t} < t - K) \\ &\quad \geq P^{\zeta,\eta}_{y-vt}(H_y \leq t) - 2P^{\zeta,\eta}_{y-vt}(H_y \leq t - L), \end{aligned} \tag{6.21}$$

where the last inequality follows from Lemma 6.3. This can be written as

$$P^{\zeta,\eta}_{y-vt}\big(H_y \in [t - L, t]\big) - P^{\zeta,\eta}_{y-vt}(H_y \leq t - L) \geq \frac{2}{3} P^{\zeta,\eta}_{y-vt}\big(H_y \in [t - L, t]\big), \tag{6.22}$$

where the last inequality is a direct consequence of Lemma 6.4. Now combining (6.21) and (6.22) we obtain (6.18), which completes the proof. □

It remains to provide the proofs of Lemmas 6.3 and 6.4.

PROOF OF LEMMA 6.3. Using the tower property for conditional expectations, we obtain

$$\begin{aligned} P^{\zeta,\eta}_{y-vt}(H_y < t - L) &\geq P^{\zeta,\eta}_{y-vt}(H_y < t - L, \mathcal{T}_{y,t} \leq t - K) \\ &= E^{\zeta,\eta}_{y-vt}\big[\mathbf{1}_{\{\mathcal{T}_{y,t} \leq t-K\}} P^{\zeta,\eta}_{y-vt}(H_y < t - L | \mathcal{F}_{\mathcal{T}_{y,t}})\big], \end{aligned} \tag{6.23}$$

where $\mathcal{F}_{\mathcal{T}_{y,t}}$ is the canonical stopped σ-algebra associated to $\mathcal{T}_{y,t}$. It follows from Lemma 4.3 that the drift of X under the tilted measure $P^{\zeta,\eta}_{y-vt}$ is always larger than $\sqrt{2|\eta|}$. On the event $\{0 \leq \mathcal{T}_{y,t} \leq t-K\}$, by the strong Markov property at time $\mathcal{T}_{y,t}$ and using that $X_{\mathcal{T}_{y,t}} = \beta_{y,t}(\mathcal{T}_{y,t})$, it holds that

$$
\begin{aligned}
P^{\zeta,\eta}_{y-vt}(H_y < t-L|\mathcal{F}_{\mathcal{T}_{y,t}}) &= P^{\zeta,\eta}_{X_{\mathcal{T}_{y,t}}}(H_y < t-L-\mathcal{T}_{y,t}) \\
&\geq \inf_{0\leq u\leq t-K} P^{\zeta,\eta}_{\beta_{y,t}(u)}(H_y \leq t-u-L) \qquad (6.24)\\
&\geq \inf_{0\leq u\leq t-K} P^{\sqrt{2|\eta|}}_{\beta_{y,t}(u)}(H_y \leq t-u-L).
\end{aligned}
$$

Recalling the assumptions of the lemma, for $u \in [0, t-K]$ we have that

$$
\begin{aligned}
E^{\sqrt{2|\eta|}}_{\beta_{y,t}(u)}(X_{t-u-L}) &= \beta_{y,t}(u) + \sqrt{2|\eta|}(t-u-L) \\
&\geq y - v_1(t-u) + v_1\left(1+\frac{2L}{K}\right)(t-u-L) \qquad (6.25)\\
&\geq y - v_1 L + v_1\frac{2L}{K}(K-L) \geq y + \frac{1}{3}v_1 L \geq y,
\end{aligned}
$$

where for the penultimate inequality we used $K-L \geq \frac{2}{3}K$, which holds by assumption. In combination with the fact that X is a Brownian motion with drift under $P^{\sqrt{2|\eta|}}_{\beta_{y,t}(u)}$, it follows that the probability on the right-hand side of (6.24) is at least $1/2$. Plugging this back into (6.23), we arrive at

$$
\begin{aligned}
P^{\zeta,\eta}_{y-vt}(H_y < t-L) &\geq \frac{1}{2}P^{\zeta,\eta}_{y-vt}(\mathcal{T}_{y,t} \leq t-K) \\
&\geq \frac{1}{2}P^{\zeta,\eta}_{y-vt}(\mathcal{T}_{y,t} \leq t-K, H_y \leq t),
\end{aligned}
$$

as claimed. □

Next, we give the proof of Lemma 6.4.

PROOF OF LEMMA 6.4. The first part of the proof of this lemma follows the same steps as the proof of Proposition 3.5 of [20] (see also the proof of Lemma 5.5). By Lemma 5.3(a), $\mathbb{P}$-a.s. for all t large enough, and all $|y| \leq 2vt$, there exist constants $\eta^{\zeta}_{y-vt,y}(v)$ so that

$$
E^{\zeta,\eta^{\zeta}_{y-vt,y}(v)}_{y-vt}[H_y] = t. \qquad (6.26)
$$

To simplify the notation, we write $\widetilde{\eta} := \eta^{\zeta}_{y-vt,y}(v)$. Using Lemma 5.3(b), we can assume that $\widetilde{\eta} < \overline{\eta}(v) + \frac{c}{2L}$, and thus, by the hypothesis of the lemma,

$$
\eta - \widetilde{\eta} > \frac{c}{2L}. \qquad (6.27)
$$

By definition of the tilted measures (cf. (4.4)),

$$
\begin{aligned}
&P^{\zeta,\eta}_{y-vt}(H_y \leq t-L) \\
&\quad = \frac{1}{Z^{\zeta,\eta}_{y-vt,y}} E_{y-vt}\big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}; H_y \leq t-L\big] \qquad (6.28)
\end{aligned}
$$

$$= \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} \frac{1}{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}} E_{y-vt}\big[e^{\int_0^{H_y}(\zeta(X_s)+\widetilde{\eta})\,\mathrm{d}s} e^{-H_y(\widetilde{\eta}-\eta)}; H_y \le t-L\big].$$

$$= \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} E^{\zeta,\widetilde{\eta}}_{y-vt}\big[e^{-H_y(\widetilde{\eta}-\eta)}; H_y \le t-L\big].$$

Define random variables $\tau_i = H_{y-vt+i} - H_{y-vt+i-1}$, $i = 1, \dots, \lfloor vt \rfloor$, and $\tau_{vt} = H_y - H_{y-vt+\lfloor vt \rfloor}$, so that $\sum_{i=1}^{\lfloor vt \rfloor} \tau_i + \tau_{vt} = H_y$, and their centred versions $\widehat{\tau}_i = \tau_i - E^{\zeta,\widetilde{\eta}}_{y-vt}[\tau_i]$ for $i = 1, \dots, \lfloor vt \rfloor$, and $\widehat{\tau}_{vt} = \tau_{vt} - E^{\zeta,\widetilde{\eta}}_{y-vt}[\tau_{vt}]$. Further, let

$$Y^{\zeta}_{y-vt,y} := \frac{(\widetilde{\eta}-\eta)}{\widetilde{\sigma}} \left(\sum_{i=1}^{\lfloor vt \rfloor} \widehat{\tau}_i + \widehat{\tau}_{vt} \right), \tag{6.29}$$

where

$$\widetilde{\sigma} = \widetilde{\sigma}^{\zeta}_{y-vt,y}(v) = |\widetilde{\eta}-\eta| \sqrt{\operatorname{Var}_{P^{\zeta,\widetilde{\eta}}_{y-vt}}(H_y)} \tag{6.30}$$

is chosen so that the variance of $Y^{\zeta}_{y-vt,y}$ is one. Denoting by $\mu^{\zeta}_{y-vt,y}$ the distribution of $Y^{\zeta}_{y-vt,y}$ under $P^{\zeta,\widetilde{\eta}}_{y-vt,y}$, using also the fact that $E^{\zeta,\widetilde{\eta}}_{y-vt,y}[H_y] = t$, by the definition of $\widetilde{\eta}$, (6.28) can be rewritten as

$$\begin{aligned} &P^{\zeta,\widetilde{\eta}}_{y-vt}(H_y \le t-L) \\ &\quad = \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} e^{(\eta-\widetilde{\eta})t} E^{\zeta,\widetilde{\eta}}_{y-vt}\left[e^{-\widetilde{\sigma} Y^{\zeta}_{y-vt,y}}; Y^{\zeta}_{y-vt,y} \in \left[\frac{L(\eta-\widetilde{\eta})}{\widetilde{\sigma}}, \infty\right)\right] \\ &\quad = \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} e^{(\eta-\widetilde{\eta})t} \int_{L(\eta-\widetilde{\eta})/\widetilde{\sigma}}^{\infty} e^{-\widetilde{\sigma} u} \mu^{\zeta}_{y-vt,y}(\mathrm{d}u). \end{aligned} \tag{6.31}$$

Setting $L = 0$ in the above formula, we further obtain

$$P^{\zeta,\eta}_{y-vt}(H_y \le t) = \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} e^{(\eta-\widetilde{\eta})t} \int_0^{\infty} e^{-\widetilde{\sigma} u} \mu^{\zeta}_{y-vt,y}(\mathrm{d}u). \tag{6.32}$$

Hence, to complete the proof of the lemma, it suffices to show that the integral on the right-hand side of (6.31) is at most 1/4 of the integral on the right-hand side of (6.32).

To see this, we proceed as in the proof of Lemma 3.6 of [20]. By the strong Markov property, the random variables $\widehat{\tau}_i$, $i = 1, \dots, \lfloor vt \rfloor$ and $\widehat{\tau}_{vt}$ are independent under $P^{\zeta,\widetilde{\eta}}_{y-vt}$. Further, it is a straightforward consequence of the definitions of the logarithmic moment generating functions in (5.2) and their being well-defined for $\eta < 0$ that these random variables have uniform exponential moments. Moreover, recall that $\widetilde{\sigma}$ was chosen such that the variance of $\mu^{\zeta}_{y-vt,y}$ is one. This allows the application of a local central limit theorem for normalised sums of independent random variables [8], Theorem 13.3, which implies that

$$\sup_B \big|\mu^{\zeta}_{y-vt,y}(B) - \Phi(B)\big| \le C_{17}\big(\lceil vt \rceil\big)^{-1/2}, \tag{6.33}$$

where the supremum is taken over all intervals B in $\mathbb{R}$ and Φ denotes the standard Gaussian measure. Note that the constant C_{17} in the last display depends only on the uniform bound of the exponential moments of the $\widehat{\tau}_i$s. Without loss of generality, we can assume that $C_{17} > 4$.

Similarly as in (5.17) (see also [20], (3.8)), the variance $\widetilde{\sigma}^2$ defined in (6.30) satisfies for $\mathbb{P}$-a.e. ζ and t large enough

$$C_{18}{}^{-1}\sqrt{\lceil vt\rceil} \le \widetilde{\sigma} \le C_{18}\sqrt{\lceil vt\rceil}. \tag{6.34}$$

We now have all ingredients to complete the proof. To this end, we assume that the constant c from the statement of the lemma satisfies the inequality

$$\ell := \frac{L(\eta-\widetilde{\eta})}{\widetilde{\sigma}} \ge \frac{c}{2C_{18}\sqrt{vt}} \ge \frac{20c_1}{\sqrt{vt}}. \tag{6.35}$$

To bound the integral in (6.31) from above, we observe that for any interval (a,b) of length ℓ we have $\Phi((a,b)) \le \ell/\sqrt{2\pi}$, and thus $\mu^{\zeta}_{y-vt,y}((a,b)) \le (\ell + C_{17}/\sqrt{vt}) \le 2\ell$, by (6.35). Therefore, using (6.34) in the last step,

$$\begin{aligned}\int_{L(\eta-\widetilde{\eta})/\widetilde{\sigma}}^{\infty} e^{-\widetilde{\sigma}u}\mu^{\zeta}_{y-vt,y}(\mathrm{d}u) &\le \sum_{i=1}^{\infty} e^{-\widetilde{\sigma}i\ell}\mu^{\zeta}_{y-vt,y}\big((i\ell,(i+1)\ell)\big)\\ &\le \frac{2\ell e^{-\widetilde{\sigma}\ell}}{1-e^{-\widetilde{\sigma}\ell}} \le \frac{2\widetilde{\sigma}\ell e^{-\widetilde{\sigma}\ell}}{1-e^{-\widetilde{\sigma}\ell}}\cdot\frac{C_{18}}{\sqrt{vt}}.\end{aligned} \tag{6.36}$$

On the other hand, using the rough bound $\Phi((0,x)) \ge x/5$, which holds for small enough x, and (6.35),

$$\begin{aligned}\int_0^{\infty} e^{-\widetilde{\sigma}u}\mu^{\zeta}_{y-vt,y}(\mathrm{d}u) &\ge \int_0^{L(\eta-\widetilde{\eta})/2\widetilde{\sigma}} e^{-\widetilde{\sigma}u}\mu^{\zeta}_{y-vt,y}(\mathrm{d}u)\\ &\ge e^{-\widetilde{\sigma}\ell/2}\mu^{\zeta}_{y-vt,y}\big((0,\ell/2)\big) \ge e^{-\widetilde{\sigma}\ell/2}\Big(\Phi\big((0,\ell/2)\big) - \frac{C_{17}}{\sqrt{vt}}\Big)\\ &\ge e^{-\widetilde{\sigma}\ell/2}\frac{c_1}{\sqrt{vt}}.\end{aligned} \tag{6.37}$$

By increasing the value of the constant c, and thus of $\widetilde{\sigma}\ell \ge c/2$, the right-hand side of (6.36) can be made at most $1/4$ times as large as the right-hand side of (6.37). This completes the proof of the lemma. □

7. Proof of the tightness of the maximum of the BBMRE. We are now ready to prove the main theorem of this paper.

PROOF OF THEOREM 2.1. For $\varepsilon \in (0,1/2)$, let $x_t = x_t(\varepsilon) \in \mathbb{R}$ the unique location where

$$w^{x_t}(t,0) = \mathrm{P}^{\xi}_0\big(M(t) \ge x_t\big) = \varepsilon. \tag{7.1}$$

As already explained in Section 2.1, to show the tightness of the recentred maximum $M(t)$ we need to show that there exists $\Delta = \Delta(\varepsilon) < \infty$ such that for all $t > 0$ it holds that

$$w^{x_t-\Delta}(t,0) = \mathrm{P}^{\xi}_0\big(M(t) \ge x_t - \Delta\big) > 1-\varepsilon. \tag{7.2}$$

Note that (7.1) and the law of large numbers for $M(t)$ (i.e., $\lim_{t\to\infty} M(t)/t = v_0$, cf. (2.6)) imply that

$$\lim_{t\to\infty}\frac{x_t}{t} = v_0, \quad \mathbb{P}\text{-a.s.} \tag{7.3}$$

A direct consequence of (7.3) is that for t large enough, we can guarantee that x_t is large enough in order to apply Corollary 3.6 to $w^{x_t}(t,0)$. Therefore, for large enough t, we infer the existence of a $\mathbb{P}$-a.s. finite random time $T < \infty$ such that

$$w^{x_t}\big(t+t',0\big) \ge 1-\varepsilon \quad \text{for all } t' \in [T, T+1]. \tag{7.4}$$

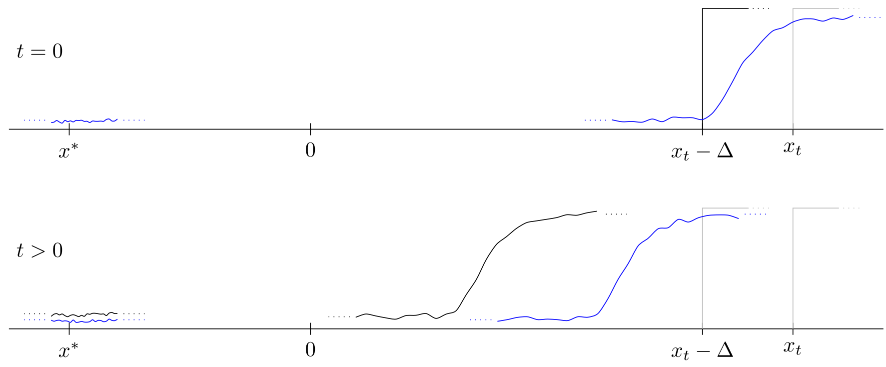

FIG. 4. *The top figure shows the graph of the function* $w^{x_t-\Delta}(0,\cdot) = \mathbf{1}_{[x_t-\Delta,\infty)}(\cdot)$ *in black and the function* $w^{x_t}(T,\cdot)$ *in blue. The lower figure shows the graph of the same functions at some positive time* $t > 0$. *By the Sturmian principle, for any* $t > 0$, *the region where* $w^{x_t-\Delta}(t,\cdot)$ *dominates* $w^{x_t}(t+T,\cdot)$ *is an interval that contains* $[x^*,\infty)$.

For any $u \in \mathbb{N}$, define the subset $\Omega_u = \{T \in [u-1,u)\}$ of the probability space on which ξ is defined. We now consider $\xi \in \Omega_u$. Observe that (7.2) would follow from (7.4) on Ω_u, if for a suitably large $\Delta < \infty$ we had

$$w^{x_t-\Delta}(t,0) \ge w^{x_t}(t+u,0). \tag{7.5}$$

Instead of comparing these two functions directly at $x = 0$, we use the Sturmian principle, to relate the inequality (7.5) at the origin to an inequality at some point on the negative half-line. More precisely, recall from Section 3.2 that for any $t > 0$, $u > 0$ and $\widetilde{\Delta} < \infty$ the difference

$$W^{u,\widetilde{\Delta}}(t,x) := w^{x_t-\widetilde{\Delta}}(t,x) - w^{x_t}(t+u,x)$$

solves a linear parabolic equation of the form (3.8), with initial condition

$$W^{u,\widetilde{\Delta}}(0,x) = \mathbf{1}_{[x_t-\widetilde{\Delta},\infty)}(x) - w^{x_t}(u,x).$$

Since $0 < w^{x_t}(u,x) < 1$ for all $u > 0$ and $x \in \mathbb{R}$ (cf. (3.10)), it follows moreover, that $W^{u,\widetilde{\Delta}}(0,x) > 0$ for $x > x_t - \widetilde{\Delta}$ and $W^{u,\widetilde{\Delta}}(0,x) < 0$ for $x < x_t - \widetilde{\Delta}$. Therefore, it holds by Lemma 3.4 that for all $t > 0$ the sets

$$\{x \in \mathbb{R} : w^{x_t-\widetilde{\Delta}}(t,x) > w^{x_t}(t+u,x)\} = \{x \in \mathbb{R} : W^{u,\widetilde{\Delta}}(t,x) > 0\}$$

are open intervals unbounded to the right. Thus, in order to prove (7.5) it suffices to find some $x^* = x^*(t) < 0$ and $\Delta < \infty$ such that $W^{u,\Delta}(t,x^*) > 0$, as this implies $0 \in \{x \in \mathbb{R} : W^{u,\Delta}(t,x) > 0\}$, which in turn implies (7.5); for an illustration of this argument, see Figure 4.

To find such $x^*(t)$ take any $v > v_2$, where v_2 is defined in (6.2). Since $v_2 > v_0$ and $(x_t - \Delta)/t \to v_0$, by (7.3), it follows that $x_t - \Delta \in [0, vt]$ for all t that are sufficiently large. Consequently, we can apply Lemma 6.1 and infer the existence of a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}(u,v)$ and some $\Delta_0(u,v) > 0$ such that if we require, additionally to the previous conditions on the size of t, that $t > \mathcal{T}(u,v)$ and that $\Delta > \Delta_0(u,v)$, then

$$w^{x_t-\Delta}(t,x^*) \ge w^{x_t}(t+u,x^*),$$

with $x^* = x_t - \Delta - vt < 0$. By the previous discussion, this implies

$$w^{x_t-\Delta}(t,0) \ge w^{x_t}(t+u,0) \ge 1-\varepsilon,$$

and hence tightness of the family $(M(t) - m^{\xi}(t))_{t\ge 0}$ for $\mathbb{P}$-a.e. $\xi \in \Omega_u$. As $\mathbb{P}(\Omega) = \mathbb{P}(\bigcup_{u\ge 1}\Omega_u) = 1$, this completes the proof. $\square$

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